Gaussian fluctuations from nonlinear SPDEs

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Problem

Interface growth:

$$\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}\beta|\nabla h|^2 + V(t,x), \quad x \in \mathbb{R}^d, d \geq 2$$

- random driving force V: white-in-time, smooth-in-space
  $$V(t,x) = \int \phi(x-y)\xi(t,y)dy,\ \phi \in C_c^\infty, \int \phi = 1$$
  ξ: spacetime white noise
- covariance function of V:
  $$\mathbb{E}[V(t,x)V(s,y)] = \delta(t-s)R(x-y)$$
  $$R(x) = \int \phi(x+y)\phi(y)dy$$
- coupling constant
  $$\beta = \hat{\beta}(|\log \varepsilon|^{-\frac{1}{2}} 1_{d=2} + 1_{d\geq 3})$$
- flat initial data $h(0,x) \equiv 0$

Question: large scale behavior of h

micro→macro

Equation: $\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}\beta|\nabla h|^2 + V(t,x), \quad x \in \mathbb{R}^d, d \geq 2$

- diffusive rescaling: $h_\varepsilon(t,x) = h(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$:

$$\partial_t h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \frac{1}{2}\beta|\nabla h_\varepsilon|^2 + \frac{1}{\varepsilon^2} V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$$

- scaling property of white noise: $\frac{1}{\varepsilon^2} V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) \overset{\text{law}}{=} \varepsilon^{\frac{d-2}{2}} \xi_\varepsilon(t,x)$

$$\xi_\varepsilon(t,x) = \int \phi_\varepsilon(x-y)\xi(t,y)dy, \qquad \phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon)$$

- small forcing in $d \geq 3$: $H_\varepsilon(t,x) = \varepsilon^{-\frac{d-2}{2}} h_\varepsilon(t,x)$:

$$\partial_t H_\varepsilon = \frac{1}{2}\Delta H_\varepsilon + \frac{1}{2}\beta_\varepsilon|\nabla H_\varepsilon|^2 + \xi_\varepsilon(t,x)$$

- coupling constant: $\beta_\varepsilon = \hat{\beta}(|\log \varepsilon|^{-\frac{1}{2}} 1_{d=2} + \varepsilon^{\frac{d-2}{2}} 1_{d\geq 3})$

Question: limit of H_ε as $\varepsilon \to 0$

Main result

Equation: $\partial_t H_\varepsilon = \frac{1}{2}\Delta H_\varepsilon + \frac{1}{2}\beta_\varepsilon |\nabla H_\varepsilon|^2 + \xi_\varepsilon(t,x), \quad H_\varepsilon(0,x) = 0$

$$\beta_\varepsilon = \hat\beta(|\log\varepsilon|^{-\frac{1}{2}} 1_{d=2} + \varepsilon^{\frac{d-2}{2}} 1_{d\geq 3})$$

Theorem (Dunlap-G.-Ryzhik-Zeitouni 18', G. 18')

There exists $\beta_0 > 0$ such that if $\hat\beta < \beta_0$, then

$$\int (H_\varepsilon(t,x) - \mathbb{E}[H_\varepsilon(t,x)])g(x)dx \Rightarrow \int \mathcal{H}(t,x)g(x)dx$$

with $\partial_t \mathcal{H} = \frac{1}{2}\Delta\mathcal{H} + \nu_{\text{eff}}\,\xi$, with $\mathcal{H}(0,x) \equiv 0$ and the effective variance

$$\nu_{\text{eff}}^2 = \begin{cases} 2\pi/(2\pi - \hat\beta^2) & d = 2 \\ \int R(x)\mathbb{E}_B[e^{\frac{1}{2}\hat\beta^2 \int_0^\infty R(x+B_s)ds}]dx & d \geq 3 \end{cases}$$

- ⋆ weak coupling/disorder $\hat\beta < \beta_0$
- ⋆ $\nu_{\text{eff}}^2 > 1$
- ⋆ Gaussian/free/trivial/Edwards-Wilkinson limit

Roadmap from KPZ to EW

- Microscopic: $\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}\beta|\nabla h|^2 + V(t,x), \quad h(0,x) = 0$
- Zoom out: $H_\varepsilon(t,x) = \varepsilon^{-\frac{d-2}{2}} h(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$
- Macroscopic: $\partial_t H_\varepsilon = \frac{1}{2}\Delta H_\varepsilon + \frac{1}{2}\beta_\varepsilon|\nabla H_\varepsilon|^2 + \xi_\varepsilon$
- Equation for fluctuations:

$$\partial_t(H_\varepsilon - \mathbb{E}[H_\varepsilon]) = \frac{1}{2}\Delta(H_\varepsilon - \mathbb{E}[H_\varepsilon]) + \frac{1}{2}\beta_\varepsilon(|\nabla H_\varepsilon|^2 - \mathbb{E}[|\nabla H_\varepsilon|^2]) + \xi_\varepsilon$$

- nonlinearity $\to$ independent noise

$$\frac{1}{2}\beta_\varepsilon(|\nabla H_\varepsilon|^2 - \mathbb{E}[|\nabla H_\varepsilon|^2]) \approx \lambda\eta_\varepsilon$$

- Edwards-Wilkinson limit $H_\varepsilon - \mathbb{E}[H_\varepsilon] \approx \mathcal{H}$

$$\partial_t(H_\varepsilon - \mathbb{E}[H_\varepsilon]) \approx \frac{1}{2}\Delta(H_\varepsilon - \mathbb{E}[H_\varepsilon]) + \lambda\,\eta_\varepsilon + \xi_\varepsilon$$

$$\partial_t\mathcal{H} = \frac{1}{2}\Delta\mathcal{H} + \nu_{\mathrm{eff}}\,\xi$$

- Effective variance: $\nu_{\mathrm{eff}}^2 = \lambda^2 + 1 > 1$

Hopf-Cole and SHE

KPZ: $\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}\beta|\nabla h|^2 + V(t,x), \quad h(0,x) = 0$

Hopf-Cole: $u(t,x) = \exp(\beta h(t,x) - \frac{1}{2}\beta^2 R(0) t)$

SHE: $\partial_t u = \frac{1}{2}\Delta u + \beta u V, \quad u(0,x) = 1$

EW: $\partial_t \mathcal{H} = \frac{1}{2}\Delta \mathcal{H} + \nu_{\mathrm{eff}}\xi$

- KPZ$\to$ EW:

$$\beta_\varepsilon^{-1}(\log u(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}) - \mathbb{E}[\log u(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon})]) \Rightarrow \mathcal{H}(t,x)$$

- SHE$\to$ EW:

$$\beta_\varepsilon^{-1}(u(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}) - 1) \Rightarrow \mathcal{H}(t,x)$$

- Linearization? $\log u = \log(1 + u - 1) \approx u - 1\,?$
- functions other than log?

$$\beta_\varepsilon^{-1}(f(u(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon})) - \mathbb{E}[f(u(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}))]) \Rightarrow ?$$

The rest of the talk

- Heuristics for SHE: Brownian paths intersection, Wiener chaos expansion
- Heuristics for KPZ: iteration and contraction
- From SHE to KPZ: analysis on Gaussian space
- Tools: Clark-Ocone formula, second order Poincaré inequality
- Nonlinear SHE
- Other works and further questions

Heuristics for SHE (I): Brownian paths intersection

Equation: $\partial_t u = \frac{1}{2}\Delta u + \beta u V, \quad u(0,x) = 1$

Scaling: $u_\varepsilon(t,x) = u(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}), \qquad \partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \beta \varepsilon^{\frac{d-2}{2}} u_\varepsilon \xi_\varepsilon$

Convergence: $\frac{u_\varepsilon - 1}{\beta \varepsilon^{(d-2)/2}} \Rightarrow \partial_t \mathcal{H} = \frac{1}{2}\Delta \mathcal{H} + \nu_{\mathrm{eff}} \xi$

- fluctuations $u_\varepsilon - 1$
  $$\partial_t \left(\frac{u_\varepsilon - 1}{\beta \varepsilon^{(d-2)/2}} \right) = \frac{1}{2}\Delta \left(\frac{u_\varepsilon - 1}{\beta \varepsilon^{(d-2)/2}} \right) + u_\varepsilon \xi_\varepsilon$$
- resonance: $u_\varepsilon \xi_\varepsilon \to \nu_{\mathrm{eff}} \xi$ with $\nu_{\mathrm{eff}} > 1$
- variance computation:
  $$\mathrm{Var}\left[\frac{1}{\beta \varepsilon^{(d-2)/2}} \int (u_\varepsilon - 1) g \right] =$$
  $$\frac{1}{\beta^2 \varepsilon^{d-2}} \int \mathrm{Cov}[u(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}), u(\tfrac{t}{\varepsilon^2}, \tfrac{y}{\varepsilon})] g(x) g(y) dx dy$$
- Feynman-Kac representation
  $$u(t,x) = \mathbb{E}_B[e^{\beta \int_0^t V(t-s, x+B_s) ds - \frac{1}{2}\beta^2 R(0) t}]$$
  $$\mathrm{Cov}[u(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}), u(\tfrac{t}{\varepsilon^2}, \tfrac{y}{\varepsilon})] = \mathbb{E}_B[e^{\beta^2 \int_0^{t/\varepsilon^2} R(\frac{x-y}{\varepsilon} + B_s^1 - B_s^2) ds}] - 1$$

Brownian paths intersection

Covariance function for SHE:

$$\mathrm{Cov}[u(\tfrac{t}{\varepsilon^2},\tfrac{x}{\varepsilon}),u(\tfrac{t}{\varepsilon^2},\tfrac{y}{\varepsilon})]=\mathbb{E}_B[e^{\beta^2\int_0^{t/\varepsilon^2}R(\frac{x-y}{\varepsilon}+B_s^1-B_s^2)ds}]-1$$

Goal: $\frac{1}{\beta^2\varepsilon^{d-2}}\mathrm{Cov}[u(\tfrac{t}{\varepsilon^2},\tfrac{x}{\varepsilon}),u(\tfrac{t}{\varepsilon^2},\tfrac{y}{\varepsilon})]\approx\mathrm{Cov}[\mathcal{H}(t,x),\mathcal{H}(t,y)]$

$$=\nu_{\mathrm{eff}}^2\int_0^t q_{2(t-s)}(x-y)ds$$

- with probability $\sim\beta^2\varepsilon^{d-2}$, two paths meet
- $\hat{\beta}$ small $\Rightarrow$ intersection time $\ll\varepsilon^{-2}$
- intersection generates ν_{eff}^2
- short intersection time $\Rightarrow$ temporospatial mixing $\Rightarrow$ Gaussianity

Question:

- $\mathrm{Cov}[\log(u(\tfrac{t}{\varepsilon^2},\tfrac{x}{\varepsilon})),\log(u(\tfrac{t}{\varepsilon^2},\tfrac{y}{\varepsilon}))]$?
- similar mixing mechanism for $\log u$?

Heuristics for SHE (II): chaos expansion

Equation: $\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \beta\varepsilon^{(d-2)/2} u_\varepsilon \xi_\varepsilon, \qquad u_\varepsilon(0,x) = 1$

Limit: $\frac{u_\varepsilon - 1}{\beta\varepsilon^{(d-2)/2}} \Rightarrow \partial_t \mathcal{H} = \frac{1}{2}\Delta\mathcal{H} + \nu_{\text{eff}}\xi$

- mild formulation:
  $u_\varepsilon(t,x) = 1 + \beta\varepsilon^{\frac{d-2}{2}} \int_0^t \int q_{t-s}(x-y) u_\varepsilon(s,y)\xi_\varepsilon(s,y)dyds$
- Wiener chaos: $\frac{1}{\beta\varepsilon^{(d-2)/2}} \int (u_\varepsilon - 1)g = \sum_{k\geq 1} I_k(f_{\varepsilon,k})$ with

$$I_k(f_{\varepsilon,k}) = \int g(x) q_{t-s_1}(x-y_1)$$

$$\xi_\varepsilon(s_1,y_1)\left[\prod_{j=2}^{k} \beta\varepsilon^{\frac{d-2}{2}} q_{s_{j-1}-s_j}(y_{j-1}-y_j)\xi_\varepsilon(s_j,y_j)\right] dydsdx$$

- $d \geq 2 \Rightarrow q_s(x)$ not square integrable at $(s,x) \approx (0,0)$
  $\Rightarrow (s_{j-1}, y_{j-1}) \approx (s_j, y_j)$
- $\xi_\varepsilon(s_1,y_1) \int \prod_{j=2}^k \beta\varepsilon^{\frac{d-2}{2}} q_{s_{j-1}-s_j}(y_{j-1}-y_j)\xi_\varepsilon(s_j,y_j) dyds$
  converges to independent spacetime white noise $\eta_k(s_1,y_1)$
- $\int g(x) q_{t-s_1}(x-y_1) \sum_{k\geq 1} \eta_k(s_1,y_1) dy_1 ds_1 dx$: chaotic CLT

Heuristics for KPZ: iteration and contraction

Equation: $\partial_t H_\varepsilon = \frac{1}{2}\Delta H_\varepsilon + \frac{1}{2}\beta_\varepsilon |\nabla H_\varepsilon|^2 + \xi_\varepsilon(t,x), \quad H_\varepsilon(0,x) = 0$

$$\beta_\varepsilon = \hat{\beta}(|\log \varepsilon|^{-\frac{1}{2}} 1_{d=2} + \varepsilon^{\frac{d-2}{2}} 1_{d\geq 3})$$

One iteration $\to$ Gaussian process: $H_1 = \mathcal{G}\xi_\varepsilon$

- covariance of $|\nabla H_1|^2$ = square of covariance of ∇H_1
- $\frac{1}{2}\beta_\varepsilon(|\nabla H_1|^2 - \mathbb{E}[|\nabla H_1|^2]) \Rightarrow$ spacetime white noise

Picard iteration: $H_n = \mathcal{G}\xi_\varepsilon + \mathcal{G}[\frac{1}{2}\beta_\varepsilon |\nabla H_{n-1}|^2]$

- $H_{n-1} - \mathbb{E}[H_{n-1}] \approx$ EW implies

  $\frac{1}{2}\beta_\varepsilon(|\nabla H_{n-1}|^2 - \mathbb{E}[|\nabla H_{n-1}|^2]) \approx$ spacetime white noise
- $\hat{\beta} \ll 1$: $H_n - \mathbb{E}[H_n] \approx$ EW with a “smaller error”
- $H_n - \mathbb{E}[H_n] \to$ EW as $n \to \infty$

From SHE to KPZ: Gaussian analysis

SHE: $\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \beta_\varepsilon u_\varepsilon \xi_\varepsilon, \quad u_\varepsilon(0,x) = 1$

Hopf-Cole: $H_\varepsilon = \beta_\varepsilon^{-1} \log u_\varepsilon$

KPZ: $\partial_t H_\varepsilon = \frac{1}{2}\Delta H_\varepsilon + \frac{1}{2}\beta_\varepsilon |\nabla H_\varepsilon|^2 + \xi_\varepsilon, \quad H_\varepsilon(0,x) = 0$

- u_ε: smooth nonlinear functional of ξ
- $\xi \xrightarrow[\text{Wiener chaos}]{\text{Feynman-Kac}} u_\varepsilon \xrightarrow{\text{log transform}} H_\varepsilon$
- H_ε: smooth nonlinear functional of ξ
- Dependence on ξ: $DH_\varepsilon = (\beta_\varepsilon u_\varepsilon)^{-1} Du_\varepsilon$

Idea: analyze $X_\varepsilon = \int (H_\varepsilon - \mathbb{E}[H_\varepsilon])g$ by tools in Malliavin calculus

- variance convergence $\mathrm{Var}[X_\varepsilon] \to \mathrm{Var}[\int \mathcal{H} g]$
- Gaussianity $\frac{X_\varepsilon}{\sqrt{\mathrm{Var}[X_\varepsilon]}} \Rightarrow N(0,1)$

Variance computation: Clark-Ocone formula

Goal: compute $\mathrm{Var}[\int (H_\varepsilon - \mathbb{E}[H_\varepsilon])g]$

Idea: no mild formulation for H_ε, resorting to Clark-Ocone!

$$H_\varepsilon(t,x) - \mathbb{E}[H_\varepsilon(t,x)] = \int_0^t \int_{\mathbb{R}^d} \mathbb{E}[D_{s,y} H_\varepsilon(t,x)|\mathcal{F}_s]\xi(s,y)dyds$$

- $H_\varepsilon(t,x) = \beta_\varepsilon^{-1} \log u_\varepsilon(t,x)$

  $$u_\varepsilon(t,x) = \mathbb{E}_B[e^{\beta_\varepsilon \int_0^t \xi_\varepsilon(t-s, x+B_s)ds - \frac{1}{2}\beta^2 R(0)\frac{t}{\varepsilon^2}}] \propto \mathbb{E}_B[e^{\beta_\varepsilon \langle \Phi, \xi \rangle}]$$

  $$\Phi = \phi_\varepsilon(x - y + B_{t-s})$$

- $$\begin{aligned} D_{s,y} H_\varepsilon(t,x) =& \beta_\varepsilon^{-1} u_\varepsilon(t,x)^{-1} D_{s,y} u_\varepsilon(t,x) \\ =& \frac{\mathbb{E}_B[e^{\beta_\varepsilon \langle \Phi, \xi \rangle} \Phi(s,y,t,x)]}{\mathbb{E}_B[e^{\beta_\varepsilon \langle \Phi, \xi \rangle}]} \end{aligned}$$

  $D_{s,y} H_\varepsilon(t,x)$: "transition kernel" from $(s,y) \mapsto (t,x)$
- Question: how to deal with u_ε^{-1} and the conditional expectation $|\mathcal{F}_s$?

Polymer paths

Malliavin derivative:

$$D_{s,y}H_\varepsilon(t,x) = \beta_\varepsilon^{-1} u_\varepsilon(t,x)^{-1} D_{s,y} u_\varepsilon(t,x)$$
$$= \frac{\mathbb{E}_B[e^{\beta_\varepsilon \langle \Phi, \xi \rangle} \Phi(s,y,t,x)]}{\mathbb{E}_B[e^{\beta_\varepsilon \langle \Phi, \xi \rangle}]}$$

Weak disorder $\hat{\beta} \ll 1$

- no intermittency!
- $u_\varepsilon(t,x)$ depends on $\{\xi(s,\cdot), s \in [t - o(1), t]\}$ approximately
- $u_\varepsilon(t,x)^{-1}$ almost independent of $\{\xi(s,\cdot), s < t - o(1)\}$
- the layer $[t - o(1), t]$ does not affect the polymer path asymptotically

Gaussianity: second order Poincaré inequality

Variance estimate: Gaussian-Poincaré inquality

- $\mathrm{Var}[X] \leq \mathbb{E}[\|DX\|^2]$

Gaussianity: Stein meets Malliavin

- X: smooth functional on Gaussian space $\mathbb{E}[X] = 0, \mathbb{E}[X^2] = 1$
- [Chatterjee 09] [Nourdin-Peccati-Reinert 09]

  $d_{TV}(X, N(0,1)) \lesssim \mathbb{E}[\|DX\|^4]^{1/4}\mathbb{E}[\|D^2 X\|^4]^{1/4}$

Our setting:

- $$D_{s,y} H_\varepsilon(t,x) = \frac{\mathbb{E}_B[e^{\beta_\varepsilon \langle \Phi, \xi \rangle} \Phi(s,y,t,x)]}{\mathbb{E}_B[e^{\beta_\varepsilon \langle \Phi, \xi \rangle}]}$$
- $D^2 H$: similar Feynman-Kac representation
- $\mathbb{E}[\|DX\|^4], \mathbb{E}[\|D^2 X\|^4]$: intersection of multiple polymer paths

No Hopf-Cole?

- Quadratic nonlinearity $\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}\beta|\nabla h|^2 + V(t,x)$
  $u = e^{\beta h}$ solves the linear equation $\partial_t u = \frac{1}{2}\Delta u + \beta u V$
  probabilistic tools include Feynman-Kac, Chaos expansion, ...
- General nonlinearity $\partial_t h = \frac{1}{2}\Delta h + \beta\Phi(\nabla h) + V(t,x)$
  simple example $\Phi(\nabla h) = |\nabla h|^4$, no results! no tools?
- The same EW limit should hold in a critical scaling regime for small β and a large class of Φ. Hairer-Quastel universality in the critical setting?
- Nonlinear SHE bridges the difficulty ?

$$\partial_t u = \frac{1}{2}\Delta u + \beta\sigma(u)V, \qquad u(0,x) \equiv 1$$

Question: fluctuations of $u(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$?

- Mild formulation saves the day:
  $u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y)\sigma(u(s,y))V(s,y)dyds$

Another problem

$$\partial_t u = \frac{1}{2}\Delta u + \beta\sigma(u)V, \qquad u(0,x) \equiv 1$$

- Fix $t > 0$, mixing property of $u(t,\cdot)$? Ergodicity and CLT
- Results on stochastic heat/wave equation/general dimensions/general noises/functional of u/general initial data: Chen, Delgado-Vences, Huang, Khoshnevisan, Nualart, Pu, Viitasaari, Zheng: $\frac{1}{N^{d/2}}\int_{|x|<N}(u(t,x)-1)dx \Rightarrow N(0,\nu^2)$
- I prefer ε to N

$$\frac{1}{N^{d/2}}\int_{|x|<N}(u(t,x)-1)dx = N^{d/2}\int_{|x|<1}(u(t,Nx)-1)dx$$
$$= \frac{1}{\varepsilon^{d/2}}\int [u(t,\tfrac{x}{\varepsilon})-1]g(x)dx$$

$$\varepsilon = \frac{1}{N}, \qquad g(x) = 1_{|x|<1}$$

- White noise limit of $\frac{1}{\varepsilon^{d/2}}[u(t,\frac{\cdot}{\varepsilon})-1]$ as random Schwartz distribution

Connection to our problem

$$\partial_t u = \frac{1}{2}\Delta u + \beta\sigma(u)V, \qquad u(0,x) \equiv 1$$

- $\frac{1}{\varepsilon^{d/2}}[u(t, \frac{\cdot}{\varepsilon}) - 1] \Rightarrow$ white noise
  $\frac{1}{\varepsilon^{d/2-1}}[u(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) - 1] \Rightarrow$ Edwards-Wilkinson
- To consider large time, we need to "avoid" intermittency: $d \geq 3, \beta \ll 1$ or $d = 2, \beta = \hat{\beta}|\log\varepsilon|^{-1/2}$ with $\hat{\beta} \ll 1$
- Gaussianity: follow [Huang-Nualart-Viitasaari 18]
- Convergence of variance: need the marginal distribution of $u(t,x)$ as $t \to \infty$
- $d \geq 3, \beta \ll 1$: [G.-Li 19]
  - convergence to stationary distribution
    $u(t,\cdot) \Rightarrow Z(\cdot)$ in distribution in $C(\mathbb{R}^d)$
  - $\frac{1}{\varepsilon^{d/2-1}}[u(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) - 1] \Rightarrow$ Edwards-Wilkinson
- Similar results in $d = 2$: [Dunlap-G. 20] limiting marginal distribution solves a forward-backward SDE

Convergence to stationary distribution

$$\partial_t u = \frac{1}{2}\Delta u + \beta\sigma(u)V, \qquad u(0,x) \equiv 1, \quad d \geq 3, \beta \ll 1$$

- Existence of invariant distribution is well-known [Da Prato-Gatarek-Zabczyk 92, Tessitore-Zabczyk 98]
- I am not able to find a proof of convergence to invariant distributions in literature
- $\partial_t u_T(t,x) = \frac{1}{2}\Delta u_T(t,x) + \beta\sigma(u_T(t,x))V(t,x), \quad t > -T$
  initial data at $-T$: $u_T(-T,x) \equiv 1$
  $u(T,\cdot) \overset{\text{law}}{=} u_T(0,\cdot)$
- $T \to \infty$: $u(T,\cdot)$ converges weakly but $u_T(0,\cdot)$ converges strongly
- $\{u_T(0,x)\}_{T\geq 0}$ is a Cauchy sequence in $L^2(\Omega)$ plus tightness in $C(\mathbb{R}^d)$ $\quad \Rightarrow$ weak convergene of $u(T,\cdot)$ in $C(\mathbb{R}^d)$

Other related works

$d \geq 3$

- [Magnen-Unterberger 17]: ξ_ε smooth in time and space
  KPZ $\approx \partial_t \mathcal{H} = \frac{1}{2}\nabla \cdot a_{\mathrm{eff}} \nabla \mathcal{H} + \nu_{\mathrm{eff}} \xi$
- [G.-Ryzhik-Zeitouni 17]: same result for SHE
- [Mukherjee-Shamov-Zeitouni 16] [Comets-Cosco-Mukherjee 18] [Cosco-Nakajima-Nakashima 20] [Lygkonis-Zygouras 20]: pointwise fluctuations of $u, \log u$, phase transition in $\hat{\beta}$, covering all $\hat{\beta} \in (0, \beta_{\mathrm{critical}})$

$d = 2$

- [Chatterjee-Dunlap 18]: tightness of $\{\mathcal{H}_\varepsilon\}_\varepsilon$ for $\hat{\beta} \ll 1$
- [Caravenna-Sun-Zygouras 18]: same KPZ result for $\hat{\beta} \in (0, \sqrt{2\pi})$
- [Caravenna-Sun-Zygouras 15]: same SHE result for $\hat{\beta} \in (0, \sqrt{2\pi})$, pointwise fluctuations of $u, \log u$, phase transition in $\hat{\beta}$

1d KPZ: Hairer, Gubinelli-Imkeller-Perkowski, Kupiainen,...

Conclusion

Summary:

- Gaussian fluctuations from KPZ equation and nonlinear SHE in a weak disorder regime (aka no intermittency)
- Tools from Gaussian analysis

Further questions:

- PDE question: without Hopf-Cole? more general nonlinearity? Hairer-Quastel type universality? [Hairer-Xu 16,18, Shen-Xu 16, Gubinelli-Perkowski 16, Furlan-Gubinelli 17]
  $\partial_t h = \Delta h + \phi(\nabla h) + V$
- Statistical physics question: what happens at $\hat{\beta}_{\mathrm{critical}}$? non-Gaussian behavior? How to "enter" intermittency regime?

  2d SHE at $\hat{\beta}_{\mathrm{critical}} = \sqrt{2\pi}$: $\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \sqrt{\frac{2\pi}{|\log \varepsilon|}} u_\varepsilon \xi_\varepsilon$
  [Bertini-Cancrini 98, Caravenna-Sun-Zygouras 18, G.-Quastel-Tsai 19]

References

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Thank you for your attention!