Grassmannian Brownian motion and related stochastic processes

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Based on joint works with F. Baudoin

Levy area and stochastic windings

Let (B_t^1, B_t^2), $t \geq 0$ be a planar B.M.

- Lévy area:

$$a(t) := \frac{1}{2}\int_0^t B_1(s)dB_2(s) - B_2(s)dB_1(s)$$
$$= \int_{B[0,t]} \alpha$$

  where $\alpha = \frac{1}{2}(xdy - ydx)$ is the area form.

- Characterization of the law: Lévy 1940, Kac 1951,Yor 80's
- Application: Gaveau 1976
- Winding number:

$$\theta_t := \int_{B[0,t]} \theta$$

  where $\theta = \frac{\alpha}{x^2+y^2} = \frac{xdy-ydx}{x^2+y^2}$ is the winding form.

B.M. on the Heisenberg group

- The first Heisenberg group **H** can be identified with $(\mathbb{R}^3, \star)$ such that

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')).$$

Left invariant vector fileds:

$$X = (1, 0, -\frac{1}{2}y)^T, \quad Y = (0, 1, \frac{1}{2}x)^T, \quad Z = (0, 0, 1)$$

The horizontal B.M. $\xi(t)$ satisfies

$$d\xi(t) = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2}y(t) \end{pmatrix} \circ dB_1(t) + \begin{pmatrix} 0 \\ 1 \\ x(t) \end{pmatrix} \circ dB_2(t)$$

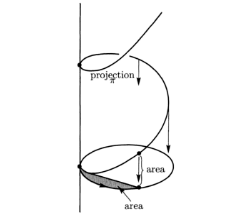
B.M. on the Heisenberg group

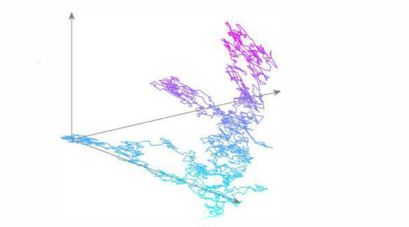
- The horizontal B.M. $\xi(t)$ started from $(0,0,0)$ is then given by

$$\xi(t) = (B_1(t), B_2(t), a(t))$$

- Fibration structure:

$$\mathbf{H} \to \mathbb{R}^2$$





Right hand side: BM on the Heisenberg group, drawn by IGL student Z. Hu

Other fibration structures

- $\mathbb{CP}^n$: set of complex lines in $\mathbb{C}^{n+1}$, $\forall z \in \mathbb{C}^{n+1}$,

$$p : (z_1, \ldots, z_{n+1})^T \mapsto (w_1, \ldots w_n)^T, \quad w_k = \frac{z_k}{z_{n+1}}.$$

- Hopf fibration

$$\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{CP}^n$$
$$(w, \theta) \mapsto \frac{e^{i\theta}}{\sqrt{1+|w|^2}}(w, 1)$$

- Let α be the area form on $\mathbb{CP}^n$:

$$\alpha = \frac{i}{2(1+|w|^2)} \sum_{j=1}^{n} (w_j d\overline{w}_j - \overline{w}_j dw_j)$$

The horizontal distribution is the kernel of $d\theta + \alpha$.

Hopf fibration

Let $w(t)$, $t \geq 0$ be a Brownian motion process on $\mathbb{CP}^n$, then the stochastic area process

$$\int_{w[0,t]} \alpha$$

- It gives the fiber motion on $\mathbb{S}^1$:

$$\begin{aligned} e^{i\theta(t)} \circ de^{-i\theta(t)} &= -d\theta(t) \\ &= \frac{i}{2(1+|w(t)|^2)} \sum_{j=1}^{n} (w_j(t) d\overline{w}_j(t) - \overline{w}_j(t) dw_j(t)) \end{aligned}$$

- [Baudoin, W. 2017] A $\mathbb{S}^{2n+1}$-valued B.M. has skew-product decomposition

$$\frac{e^{iB(t)-i\theta(t)}}{\sqrt{1+|w(t)|^2}}(w(t),1), \quad t \geq 0$$

where $B(t)$, $t \geq 0$ is a B.M. indep. of w.

Hopf fibration

Let $r = \arctan|w|$, then r gives the Riemannian distance on $\mathbb{CP}^n$.

- $(r(t), \theta(t))$, $t \geq 0$ is the diffusion generated by

$$\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + ((2n-1)\cot r - \tan r)\frac{\partial}{\partial r} + \tan^2 r \frac{\partial^2}{\partial \theta^2}\right)$$

hence

$$(r(t), \theta(t))_{t\geq 0} \overset{d}{=} \left(r(t), B_{\int_0^t \tan^2 r(s) ds}\right)_{t\geq 0},$$

- [Baudoin-W.,2017]

$$\mathbb{E}\left(e^{i\lambda\theta(t)}\right) = e^{-n|\lambda|t}\int_0^{\pi/2} \frac{q_t^{n-1,|\lambda|}(0,r)}{(\cos r)^{|\lambda|}} dr$$

As $t \to \infty$

$$\frac{\theta(t)}{t} \to \text{Cauchy}(n).$$

Stiefel fibration

- The complex Stiefel manifold $V_{n,k}$: set of k-frames in $\mathbb{C}^n$, i.e., first k columns of a unitary matrix.

  e.g. $V_{n,1}$ can be identified as $\mathbb{S}^{2n-1}$.

- The Grassmannian manifold $G_{n,k}$: k-dim subspace of $\mathbb{C}^n$, i.e.

$$G_{n,k} = V_{n,k}/U(k)$$

  e.g. $G_{n,1}$ can be identified as $\mathbb{CP}^n$.

- The Stiefel fibration

$$U(k) \to V_{n,k} \to G_{n,k}.$$

  $k = 1$ gives the Hopf fibration.

Matrix presentation

Let

$$U = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

Exclude $\det(Z) = 0$ and consider

$$\begin{aligned} p : \hat{V}_{n,k} &\longrightarrow \mathbb{C}^{(n-k)\times k} \cong \hat{G}_{n,k} \\ \begin{pmatrix} X \\ Z \end{pmatrix} &\longmapsto XZ^{-1} \end{aligned}$$

Let $U(t)$, $t \geq 0$ be a B.M. on $U(n)$. It solves the SDE

$$dU(t) = U(t) \circ dA(t)$$

where $A(t)$ is a Brownian motion on $\mathfrak{u}(n)$.

Stiefel Brownian motion

Theorem (Baudoin-W. 2020)

- *The process* $(X(t)\ Z(t))^T$ *is a B.M. on* $V_{n,k}$.
-
$$\mathbb{P}\left(\inf\{t > 0, \det Z(t) = 0\} < +\infty\right) = 0.$$
- *The process* $w(t) := X(t)Z^{-1}(t)$, $t \geq 0$ *is a B.M. on* $\hat{G}_{n,k}$, *with generator*

$$\Delta_{\widehat{G}_{n,k}} = 4 \sum_{1 \leq i,i' \leq n-k, 1 \leq j,j' \leq k} (I_{n-k} + ww^*)_{ii'}(I_k + w^* w)_{j'j} \frac{\partial^2}{\partial w_{ij} \partial \bar{w}_{i'j'}}.$$

The J process and the eigenvalue process

Let $J(t) := w^*(t)w(t)$.

- Then $\mathbb{P}(\inf\{t > 0, \det(J_t) = 0\} < +\infty) = 0$.

$$dJ = \sqrt{I_k + J}dB^*\sqrt{I_k + J}\sqrt{J} + \sqrt{J}\sqrt{I_k + J}dB\sqrt{I_k + J}$$
$$+ 2(n - k + \mathrm{tr}(J))(I_k + J)dt$$

Let $\lambda(t) = (\lambda_1(t), \dots, \lambda_k(t))$, $t \geq 0$ be the eigenvalue process, and assume $\lambda_1(0) > \cdots > \lambda_k(0)$,

- Then

$$\mathbb{P}\left(\forall\, t \geq 0, \lambda_1(t) > \cdots > \lambda_k(t)\right) = 1.$$

$$d\lambda_i = 2(1 + \lambda_i)\sqrt{\lambda_i}dB^i + 2(1 + \lambda_i)\Bigg(n - 2k + 1 - (2k - 3)\lambda_i$$
$$+ 2\lambda_i(1 + \lambda_i)\sum_{\ell \neq i}\frac{1}{\lambda_i - \lambda_\ell}\Bigg)dt, \quad i = 1, \dots, k.$$

The eigenvalue process

- Let $\rho_i = \frac{1-\lambda_i}{1+\lambda_i}$, then ρ is the eigenvalue process of $2ZZ^* - I_k$.

- $\rho(t)$, $t \geq 0$ is a Karlin-McGregor diffusion associated to a k-dim Jacobi process (with indep. comp.) conditioned by its ground state.

$$\mathcal{L}_{n,k} = 2\sum_{i=1}^{k}(1-\rho_i^2)\partial_i^2 - 2\sum_{i=1}^{k}\left(n-2k+(n-2k+2)\rho_i + 2\sum_{\ell\neq i}\frac{1-\rho_i^2}{\rho_\ell - \rho_i}\right)\partial_i.$$

- The density of $\rho(t)$, $t \geq 0$ is given by Assiots-O'Connell-Warren 2016.
- The complex matrix Jacobi processes $X^*(t)X(t)$, $Z^*(t)Z(t)$ have been studied by Grabiner, Doumerc, Graczyk-Malecki, Demni, etc

Stochastic area process

Recall Stiefel fibration

$$U(k) \to \hat{V}_{n,k} \to \hat{G}_{n,k}$$

- The $U(k)$-valued one-form

$$\alpha := \frac{1}{2}\Big((I_k + w^* w)^{-1/2}(dw^*\, w - w^* dw)(I_k + w^* w)^{-1/2} \\ -(I_k + w^* w)^{-1/2}\, d(I_k + w^* w)^{1/2} + d(I_k + w^* w)^{1/2}\,(I_k + w^* w)^{-1/2} \Big).$$

- Let $\mathfrak{a}(t) = \int_{w[0,t]} \alpha$
- The $U(k)$-valued fiber motion $(\Theta_t)_{t\geq 0}$:

$$\begin{cases} d\Theta_t = \circ d\mathfrak{a}_t\, \Theta_t \\ \Theta_0 = (Z_0 Z_0^*)^{-1/2} Z_0, \end{cases}$$

Skew-product decomposition

Theorem [Baudoin, W. 2020]

Let $(w_t)_{t\geq 0}$ be a Brownian motion on $\widehat{G}_{n,k}$ started at $w_0 \in \widehat{G}_{n,k}$, and $(\Omega_t)_{t\geq 0}$ is an independent B. M. on $U(k)$. The process

$$\begin{pmatrix} w_t \\ I_k \end{pmatrix} (I_k + w_t^* w_t)^{-1/2} \Theta_t \, \Omega_t$$

is a Brownian motion on $\widehat{V}_{n,k}$ started at $\begin{pmatrix} X_0 \\ Z_0 \end{pmatrix}$.

- When $k = 1$, a $\mathbb{S}^{2n+1}$-valued B.M. has skew-prod decomp.

$$\frac{e^{iB(t) - i\theta(t)}}{\sqrt{1 + |w(t)|^2}} \begin{pmatrix} w(t) \\ 1 \end{pmatrix}, \quad t \geq 0$$

  where $B(t)$, $t \geq 0$ is a B.M. indep. of $w(t)$.

Limiting theorem

Note

$$\mathrm{tr}(d\alpha) = \partial\overline{\partial} \log \det(I_k + w^* w),$$

thus $i\mathrm{tr}(d\alpha)$ is indeed the Kähler form on $\hat{G}_{n,k}$.

Generalized stochastic area:

$$\int_{w[0,t]} \mathrm{tr}(\alpha) = \frac{1}{2}\mathrm{tr}\left[\int_0^t (I_k + J)^{-1/2}(dw^*\, w - w^* dw)(I_k + J)^{-1/2}\right]$$

Theorem (Baudoin-W. 2020)

The following convergence holds in distribution when $t \to +\infty$

$$\frac{1}{it}\int_{w[0,t]} \mathrm{tr}(\alpha) \to \mathcal{C}_{k(n-k)},$$

where $\mathcal{C}_{k(n-k)}$ is a Cauchy distribution of parameter $k(n-k)$.

Sketch of proof

First obtain

$$\int_{w[0,t]} \mathrm{tr}(\alpha) \stackrel{d}{=} i\mathcal{B}_{\int_0^t \mathrm{tr}(J)ds}$$

where $\mathcal{B}$ is a 1-d B.M. indep. from $\mathrm{tr}(J)$. For every $\lambda > 0$,

$$\mathbb{E}\left(e^{-\lambda \frac{1}{i} \int_{w[0,t]} \mathrm{tr}(\alpha)}\right) = \mathbb{E}\left(e^{-\lambda \mathcal{B}_{\int_0^t \mathrm{tr}(J)ds}}\right) = \mathbb{E}\left(e^{-\frac{\lambda^2}{2} \int_0^t \mathrm{tr}(J)ds}\right).$$

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Secondly, compute and proof the martingale

$$\begin{aligned} M_t^\lambda &:= \exp\left(- \lambda \int_0^t \mathrm{tr}(\sqrt{J}(dB + dB^*)) - 2\lambda^2 \int_0^t \mathrm{tr}(J)ds \right) \\ &= e^{2k\lambda(n-k)t} \left(\frac{\det(I_k + J_0)}{\det(I_k + J_t)} \right)^\lambda \exp\left(-2\lambda^2 \int_0^t \mathrm{tr}(J)ds \right) \end{aligned}$$

Sketch of proof

Consider

$$P^{\lambda}|_{\mathcal{F}_t} = M_t^{\lambda} \cdot P|_{\mathcal{F}_t},$$

then

$$\mathbb{E}\left(e^{-2\lambda^2 \int_0^t \mathrm{tr}(J)ds}\right) = e^{-2k(n-k)\lambda t}\, \mathbb{E}^{\lambda}\left[\left(\frac{\det(I_k + J_t)}{\det(I_k + J_0)}\right)^{\lambda}\right].$$

Use the density formula of non-colliding Jacobi processes ρ_i, we have

$$\begin{aligned}&\mathbb{E}\left(e^{-2\lambda^2 \int_0^t \mathrm{tr}(J)ds}\right)\\ =&C_{\mathrm{ini}} e^{C_{\lambda,n,k} t} \int_{\Delta_k} \det\left(\frac{p_t^{n-2k,2\lambda}\left(\frac{1-\lambda_i(0)}{1+\lambda_i(0)}, x_j\right)}{(1+x_j)^{\lambda}}\right)_{i,j} \prod_{i>j}(x_i - x_j)\, dx.\end{aligned}$$

Lastly take $\lambda \to \lambda/t$,

$$\lim_{t\to+\infty} \mathbb{E}\left(e^{-\lambda \frac{1}{i} \int_{w[0,t]} \mathrm{tr}(\alpha)}\right) = \lim_{t\to+\infty} \mathbb{E}\left(e^{-\frac{\lambda^2}{t^2} \int_0^t \mathrm{tr}(J)ds}\right) = e^{-k(n-k)\sqrt{2\lambda}}.$$

Asymptotic windings

Consider complex valued process $\det(Z_t)$. Since

$$\det(Z_t) = \det(I_k + w_t^* w_t)^{-1/2} \det \Theta_t \ \det \Omega_t.$$

and

$$\det \Theta_t = \frac{\det Z_0}{|\det Z_0|} \exp\left(\int_{w[0,t]} \mathrm{tr}(\alpha) \right).$$

Theorem (Baudoin-W. 2020)

One has the polar decomposition

$$\det(Z_t) = \varrho_t e^{i\theta_t}$$

where $0 < \varrho_t \leq 1$ *is a continuous semimartingale and* θ_t *is a continuous martingale such that as* $t \to +\infty$

$$\frac{\theta_t}{t} \xrightarrow{d} \mathit{Cauchy}(k(n-k)).$$