Ergodicity and Gaussian fluctuations for the stochastic wave equation

David Nualart

Department of Mathematics
Kansas University

SAUCY20
September 15, 2020

Stochastic wave equation

- Consider the stochastic wave equation in dimension $d = 1, 2, 3$

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \sigma(u)\dot{W}, \ \ t > 0,$$

with initial conditions $u(0, x) = 1$ and $\frac{\partial u}{\partial x}(0, x) = 0$.

Stochastic wave equation

- Consider the stochastic wave equation in dimension $d = 1, 2, 3$

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \sigma(u)\dot{W}, \ \ t > 0,$$

with initial conditions $u(0, x) = 1$ and $\frac{\partial u}{\partial x}(0, x) = 0$.

- $\dot{W} = \{\dot{W}(t, x), t \geq 0, x \in \mathbb{R}^d\}$ is a generalized centered Gaussian random field with covariance

$$E[\dot{W}(t, x)\dot{W}(s, y)] = \delta_0(t - s)\gamma(x - y),$$

where $\gamma(x)$ is a tempered measure, whose Fourier transform μ satisfies Dalang's condition:

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty.$$

The noise

- Let $\{W(\phi), \phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ be a centered Gaussian family of random variables with covariance

$$E[W(\phi)W(\psi)] = \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} \phi(s,x)\psi(s,y)\gamma(x-y)dxdyds =: \langle \phi, \psi \rangle_{\mathfrak{H}}.$$

The noise

- Let $\{W(\phi), \phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ be a centered Gaussian family of random variables with covariance

$$E[W(\phi)W(\psi)] = \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} \phi(s,x)\psi(s,y)\gamma(x-y)dxdyds =: \langle \phi, \psi \rangle_{\mathfrak{H}}.$$

- If $\mathfrak{H}$ is the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ with the norm $\|\cdot\|_{\mathfrak{H}}$, then $\{W(\phi), \phi \in \mathfrak{H}\}$ is an isonormal Gaussian process on the Hilbert space $\mathfrak{H}$.

The noise

- Let $\{W(\phi), \phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ be a centered Gaussian family of random variables with covariance

$$E[W(\phi)W(\psi)] = \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} \phi(s,x)\psi(s,y)\gamma(x-y)dxdyds =: \langle \phi, \psi \rangle_{\mathfrak{H}}.$$

- If $\mathfrak{H}$ is the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ with the norm $\|\cdot\|_{\mathfrak{H}}$, then $\{W(\phi), \phi \in \mathfrak{H}\}$ is an isonormal Gaussian process on the Hilbert space $\mathfrak{H}$.
- Heuristically,

$$W(\phi) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \phi(t,x) W(dt,dx).$$

and $\dot{W}(t,x) = \frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}$.

Stochastic integration

- Flor all $t \geq 0$, $\mathcal{F}_t$ is the σ-algebra generated by $\{W(\phi), \phi$ has support on $[0, t] \times \mathbb{R}^d\}$.

Stochastic integration

- Flor all $t \geq 0$, $\mathcal{F}_t$ is the σ-algebra generated by $\{W(\phi), \phi$ has support on $[0, t] \times \mathbb{R}^d\}$.
- A random field $u = \{u(t, x), t \geq 0, x \in \mathbb{R}^d\}$ is adapted if $u(t, x)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$ and $x \in \mathbb{R}^d$.

Stochastic integration

- Flor all $t \geq 0$, $\mathcal{F}_t$ is the σ-algebra generated by $\{W(\phi), \phi$ has support on $[0, t] \times \mathbb{R}^d\}$.
- A random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is adapted if $u(t,x)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$ and $x \in \mathbb{R}^d$.
- We can define the stochastic integral of a measurable and adapted random field u such that $E[\|u\|_{\mathfrak{H}}^2] < \infty$ in such a way that the following isometry property is satisfied:

$$E\left(\left|\int_{\mathbb{R}_+ \times \mathbb{R}^d} u(t,x) W(dt, dx)\right|^2\right) = E[\|u\|_{\mathfrak{H}}^2].$$

Mild solution

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))W(ds,dy),$$

where

$$G_t(x) = \begin{cases} \frac{1}{2}\mathbf{1}_{\{|x|\le t\}} & \text{if } d=1 \\ \frac{1}{\sqrt{2\pi}}(t^2 - |x|^2)^{-1/2}\mathbf{1}_{\{|x|\le t\}} & \text{if } d=2 \\ \frac{1}{4\pi t}\sigma_t(dx) & \text{if } d=3, \end{cases}$$

and σ_t is the uniform measure on $\{x \in \mathbb{R}^3 : |x| = t\}$.

Mild solution

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))W(ds,dy),$$

where

$$G_t(x) = \begin{cases} \frac{1}{2}\mathbf{1}_{\{|x|\le t\}} & \text{if } d=1 \\ \frac{1}{\sqrt{2\pi}}(t^2-|x|^2)^{-1/2}\mathbf{1}_{\{|x|\le t\}} & \text{if } d=2 \\ \frac{1}{4\pi t}\sigma_t(dx) & \text{if } d=3, \end{cases}$$

and σ_t is the uniform measure on $\{x \in \mathbb{R}^3 : |x| = t\}$.

Theorem (Dalang '99)

There is a unique mild solution, which is a random field $u = \{u(t,x), t \ge 0, x \in \mathbb{R}^d\}$ *adapted to* $\mathcal{F}_t$*, such that for all* $p \ge 2$*,* $x \in \mathbb{R}^d$*,* $T > 0$ *and* $t \in [0,T]$*,*

$$\|u(t,x)\|_p < K_{T,p}.$$

Theorem (N.-Zheng '20)

(i) *Fix* $t > 0$. *Then, the random field* $\{u(t,x), x \in \mathbb{R}^d\}$ *is stationary.*

(ii) *Assume that*

$$\mu(\{0\}) = 0.$$

Then, the random field $\{u(t,x), x \in \mathbb{R}^d\}$ *is ergodic.*

Theorem (N.-Zheng '20)

(i) *Fix $t > 0$. Then, the random field $\{u(t,x), x \in \mathbb{R}^d\}$ is stationary.*

(ii) *Assume that*

$$\mu(\{0\}) = 0.$$

Then, the random field $\{u(t,x), x \in \mathbb{R}^d\}$ is ergodic.

- A similar result for the stochastic heat equation has been proved by Chen-Khoshnevisan-N.-Pu.

Theorem (N.-Zheng '20)

(i) *Fix $t > 0$. Then, the random field $\{u(t,x), x \in \mathbb{R}^d\}$ is* *stationary*.

(ii) *Assume that*

$$\mu(\{0\}) = 0.$$

Then, the random field $\{u(t,x), x \in \mathbb{R}^d\}$ is *ergodic*.

- A similar result for the stochastic heat equation has been proved by Chen-Khoshnevisan-N.-Pu.
- By the Ergodic Theorem

$$\frac{1}{R^d} \int_{B_R} [u(t,x) - 1] dx \stackrel{R\to\infty}{\longrightarrow} 0$$

a.s. and in $L^p(\Omega)$ for all $p \geq 1$, where $B_R = \{x \in \mathbb{R}^d : |x| \leq R\}$.

Proof:

(i) The stationarity is a consequence of the constant initial condition and the fact that the covariance is spatially homogeneous.

Proof:

(i) The stationarity is a consequence of the constant initial condition and the fact that the covariance is spatially homogeneous.

(ii) The L^2 von-Neuman Ergodic Theorem, implies that, as $R \to \infty$,

$$\frac{1}{R^d} \int_{B_R} e^{i \sum_{j=1}^m z_j u(t,x+\zeta^j)} dx \overset{L^2(\Omega)}{\longrightarrow} E\left[e^{i \sum_{j=1}^m z_j u(t,\zeta^j)} | \mathcal{I} \right],$$

for all $z_1, \dots, z_m \in \mathbb{R}$ and $\zeta^1, \dots, \zeta^m \in \mathbb{R}^d$ where $\mathcal{I}$ is the invariant σ-algebra of $u(t)$. Then,

$$u(t) \text{ ergodic } \Leftrightarrow \mathcal{I} \text{ trivial } \Leftrightarrow \operatorname{Var}\left(\frac{1}{R^d} \int_{B_R} e^{i \sum_{j=1}^m z_j u(t,x+\zeta^j)} dx \right) \to 0.$$

This variance can be estimated using Poincaré-type inequalities and Malliavin calculus.

Malliavin Calculus

- $\mathcal{S}$ is the space of random variables of the form

$$F = f(W(h_1), \ldots, W(h_n)),$$

where $h_i \in \mathfrak{H}$ and $f \in C_b^\infty(\mathbb{R}^n)$.

Malliavin Calculus

- $\mathcal{S}$ is the space of random variables of the form

$$F = f(W(h_1), \dots, W(h_n)),$$

  where $h_i \in \mathfrak{H}$ and $f \in C_b^\infty(\mathbb{R}^n)$.

- If $F \in \mathcal{S}$ we define its *derivative* by

$$D_{s,y}F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n))h_i(s,y).$$

  DF is a random variable with values in $\mathfrak{H}$.

- $\mathbb{D}^{1,2} \subset L^2(\Omega; \mathfrak{H})$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|DF\|_{1,2} = \sqrt{E(F^2) + E(\|DF\|_{\mathfrak{H}}^2)}.$$

- The adjoint of D is the *divergence* operator δ defined by the duality relationship

$$E(\langle DF, u\rangle_{\mathfrak{H}}) = E(F\delta(u))$$

for any $F \in \mathbb{D}^{1,2}$ and $u \in \mathrm{Dom}\delta \subset L^2(\Omega; \mathfrak{H})$.

- The adjoint of D is the *divergence* operator δ defined by the duality relationship

$$E(\langle DF, u\rangle_{\mathfrak{H}}) = E(F\delta(u))$$

for any $F \in \mathbb{D}^{1,2}$ and $u \in \mathrm{Dom}\delta \subset L^2(\Omega; \mathfrak{H})$.

- If $v \in L^2(\Omega; \mathfrak{H})$ is a square integrable and adapted, then v belongs to the domain of δ and $\delta(v)$ coincides with the Dalang-Walsh integral of v:

$$\delta(v) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} v(s, y) W(ds, dy).$$

Poincaré-type inequality

- Clark-Ocone formula: For any $F \in \mathbb{D}^{1,2}$

$$F = E[F] + \int_0^\infty \int_{\mathbb{R}^d} E[D_{s,y} F | \mathcal{F}_s] W(ds, dy).$$

Poincaré-type inequality

- Clark-Ocone formula: For any $F \in \mathbb{D}^{1,2}$

$$F = E[F] + \int_0^\infty \int_{\mathbb{R}^d} E[D_{s,y}F|\mathcal{F}_s]W(ds, dy).$$

- Poincaré inequality: For all $F, G \in \mathbb{D}^{1,2}$,

$$|\mathrm{Cov}(F, G)| \leq \int_0^\infty \int_{\mathbb{R}^{2d}} \|D_{s,y}F\|_2 \|D_{s,z}G\|_2 \gamma(y - z) dydzds,$$

provided $\|D_{s,y}F\|_2$ and $\|D_{s,z}G\|_2$ exist.

Back to the proof of ergodicity:

- Ergodicity holds it

$$A_R = \mathrm{Var}\left(\frac{1}{R^d}\int_{B_R}\prod_{j=1}^{m} g_j(u(t,x+\zeta^j))dx\right) \stackrel{R\to\infty}{\longrightarrow} 0,$$

where $g_j \in C_b^1(\mathbb{R})$ and $\zeta^j \in \mathbb{R}^d$.

Back to the proof of ergodicity:

- Ergodicity holds it

$$A_R = \mathrm{Var}\left(\frac{1}{R^d}\int_{B_R}\prod_{j=1}^m g_j(u(t,x+\zeta^j))dx\right) \stackrel{R\to\infty}{\longrightarrow} 0,$$

where $g_j \in C_b^1(\mathbb{R})$ and $\zeta^j \in \mathbb{R}^d$.

- By the Poincaré inequality

$$A_R \le \sum_{j,\ell=1}^m \frac{C}{R^{2d}}\int_{B_R^2}\int_0^\infty\int_{\mathbb{R}^{2d}} \|D_{s,y}u(t,x+\zeta^j)\|_2\|D_{s,z}u(t,x'+\zeta^\ell)\|_2 \times \gamma(y-z)dydzdxdx'ds$$

- The derivative satisfies the linear SPDE

$$D_{s,y}u(t,x) = G_{t-s}(x-y)\sigma(u(s,y)) + \int_s^t \int_{\mathbb{R}^d} G_{t-r}(x-z)\sigma'(u(r,z))D_{s,y}u(r,z)W(dr,dz).$$

- The derivative satisfies the linear SPDE

$$D_{s,y}u(t,x) = G_{t-s}(x-y)\sigma(u(s,y)) + \int_s^t \int_{\mathbb{R}^d} G_{t-r}(x-z)\sigma'(u(r,z))D_{s,y}u(r,z)W(dr,dz).$$

- We have for all $0 < s < t \le T$, $p \ge 2$,

$$\boxed{\|D_{s,y}u_{\varepsilon,k}(t,x)\|_p \le C\mathbf{1}_{\{|x-y|\le M\}}}$$

where $u_{\varepsilon,k}$ is the k-th Picard approximation of the equation

$$u_\varepsilon(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} (G_{t-s} * \psi_\varepsilon)(x-y)\sigma(u_\varepsilon(s,y))W(ds,dy),$$

where C and M depend on ε, k, p, T and the support of ψ.

- By triangular inequality, it suffices to show that for fixed ε and k,

$$K_R = \frac{1}{R^{2d}} \int_{B_R^2} \int_{\mathbb{R}^{2d}} \mathbf{1}_{\{|x-y+\zeta^j| \le M\}} \mathbf{1}_{\{|x'-z+\zeta^\ell| \le M\}} \gamma(y-z) dydzdxdx' \overset{N\to\infty}{\to} 0.$$

- By triangular inequality, it suffices to show that for fixed ε and k,

$$K_R = \frac{1}{R^{2d}} \int_{B_R^2} \int_{\mathbb{R}^{2d}} \mathbf{1}_{\{|x-y+\zeta^j|\le M\}} \mathbf{1}_{\{|x'-z+\zeta^\ell|\le M\}} \gamma(y-z) dydzdxdx' \overset{N\to\infty}{\to} 0.$$

- Using Fourier analysis

$$\begin{aligned} K_R &= \frac{1}{R^{2d}} \int_{B_R^2} \int_{\mathbb{R}^d} e^{-i(x-x'+\zeta^j-\zeta^\ell)\cdot\xi} |\mathcal{F}\mathbf{1}_{B_M}(\xi)|^2 \mu(d\xi) dxdx' \\ &\le \int_{\mathbb{R}^d} \left(\int_{B_1^2} e^{-iR(x-x')\cdot\xi} dxdx' \right) |\mathcal{F}\mathbf{1}_{B_M}(\xi)|^2 \mu(d\xi) \overset{R\to\infty}{\to} 0 \end{aligned}$$

provided $\mu(\{0\}) = 0$.

Functional Central Limit Theorem

- Suppose that $d = 1, 2$ and $\gamma(x) = |x|^{-\beta}$ (Riesz covariance), where $\beta < \min(d, 2)$.

Functional Central Limit Theorem

- Suppose that $d = 1, 2$ and $\gamma(x) = |x|^{-\beta}$ (Riesz covariance), where $\beta < \min(d, 2)$.
- When $d = 1$ and $\beta = 1$ we assume $\gamma(x) = \delta_0(x)$ (space-time white noise).

Functional Central Limit Theorem

- Suppose that $d = 1, 2$ and $\gamma(x) = |x|^{-\beta}$ (Riesz covariance), where $\beta < \min(d, 2)$.
- When $d = 1$ and $\beta = 1$ we assume $\gamma(x) = \delta_0(x)$ (space-time white noise).
- For any $t > 0$, set

$$\mathcal{S}_{R,t} = R^{-d} \int_{B_R} [u(t,x) - 1] dx.$$

Functional Central Limit Theorem

- Suppose that $d = 1, 2$ and $\gamma(x) = |x|^{-\beta}$ (Riesz covariance), where $\beta < \min(d, 2)$.
- When $d = 1$ and $\beta = 1$ we assume $\gamma(x) = \delta_0(x)$ (space-time white noise).
- For any $t > 0$, set

$$\mathcal{S}_{R,t} = R^{-d} \int_{B_R} [u(t,x) - 1] dx.$$

- Asymptotics of the variance:

$$\mathrm{Var}(\mathcal{S}_{R,t}) \sim R^{-\beta}, \quad R \to \infty$$

Theorem

As R tends to infinity,

$$R^{\frac{\beta}{2}} \mathcal{S}_{R,t} \overset{C(\mathbb{R}_+;\mathbb{R})}{\longrightarrow} \int_0^t (t-s)\xi(s) dB_s$$

where

$$\xi(s) = \sqrt{2E[\sigma^2(u(s,0))]}, \quad \text{if } \gamma = \delta_0,$$

$$\xi(s) = c_\beta E[\sigma(u(s,0))], \quad \text{otherwise},$$

and B is a Brownian motion.

Quantitative CLT

- The total variation distance between two random variables F and G is defined by

$$d_{TV}(F,G) = \sup_{B \in \mathcal{B}(\mathbb{R})} |P(F \in B) - P(G \in B)|.$$

Theorem

For every $t > 0$, there exist a constant $c(t)$, such that

$$d_{TV}\left(\frac{\mathcal{S}_{R,t}}{\sqrt{\mathrm{Var}(\mathcal{S}_{R,t})}}, Z\right) \leq c(t) R^{-\beta/2},$$

where Z has law $N(0,1)$.

- References: Delgado-H.-Zheng '20 (case $d=1$), Bolaños-N.-Zheng '20 (case $d=2$).

Stein's method for normal approximations

Lemma

A random variable Z such that $E[|Z|] < \infty$ has the $N(0,1)$ law if and only if for any $\phi \in C_b^1(\mathbb{R})$,

$$E[\phi'(Z) - Z\phi(Z)] = 0 \tag{1}$$

Stein's method for normal approximations

Lemma

A random variable Z such that $E[|Z|] < \infty$ *has the* $N(0,1)$ *law if and only if for any* $\phi \in C_b^1(\mathbb{R})$,

$$E[\phi'(Z) - Z\phi(Z)] = 0 \tag{1}$$

- Main idea: Given a random variable F, if $E[\phi'(F) - F\phi(F)]$ is small for a suitable class of functions ϕ, then the distribution of F is close to the standard normal distribution.

- Louis H. Y. Chen, Larry Goldstein and Qi-Man Shao: *Normal approximation by Stein's method*. Springer, 2011.

Stein's equation

Let $Z \sim N(0,1)$ and let $h : \mathbb{R} \to \mathbb{R}$ be such that $E(|h(Z)|) < \infty$. Stein's equation associated with h is the linear differential equation

$$\boxed{\phi_h'(x) - x\phi_h(x) = h(x) - E(h(Z))}, \quad x \in \mathbb{R}. \tag{2}$$

- The function

$$\phi_h(x) = e^{x^2/2} \int_{-\infty}^{x} (h(y) - E[h(Z)]) e^{-y^2/2} dy.$$

  is the unique solution of Stein's equation satisfying

$$\lim_{x \to \pm\infty} e^{-x^2/2} \phi_h(x) = 0. \tag{3}$$

- ϕ_h satisfies

$$\|\phi_h\|_\infty \le \sqrt{\frac{\pi}{2}} \quad \text{and} \quad \|\phi_h'\|_\infty \le 2.$$

Proposition

Let F and Z be two random variables such that $Z \sim N(0,1)$. Then,

$$d_{TV}(F,Z) \leq \sup_{\phi \in \mathscr{F}_{TV}} \left| E[\phi'(F) - F\phi(F)] \right|,$$

where

$$\mathscr{F}_{TV} = \{\phi \in \mathcal{C}^1(\mathbb{R}) : \|\phi\|_\infty \leq \sqrt{\frac{\pi}{2}}, \|\phi'\|_\infty \leq 2\}.$$

Proof:

- Let $h : \mathbb{R} \to [0,1]$ be a continuous function and let ϕ_h be the solution to the Stein's equation associated with h, that is,

$$h(x) - E[h(Z)] = \phi_h'(x) - x\phi_h(x).$$

- Replacing x by F and taking the expectation, yields

$$\begin{aligned} |E(h(F)) - E(h(Z))| &= |E[\phi_h'(F) - F\phi_h(F)]| \\ &\leq \sup_{\phi \in \mathcal{C}^1(\mathbb{R}): \|\phi\|_\infty \leq \sqrt{\frac{\pi}{2}}, \|\phi'\|_\infty \leq 2} |E[\phi'(F) - F\phi(F)]| . \end{aligned}$$

- This inequality holds for any $h : \mathbb{R} \to [0,1]$ measurable, because we can approximate h by continuous functions almost everywhere with respect to the measure $P^F + P^Z$. Taking $h = \mathbf{1}_B$, we obtain the result.

Stein meets Malliavin

Let $W = \{W(h), h \in \mathfrak{H}\}$ be an isonormal Gaussian process defined on $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is generated by W.

Theorem (Nourdin-Peccati)

Suppose that $F \in \mathbb{D}^{1,2}$ *satisfies* $E[F^2] = 1$ *and* $F = \delta(u)$*, where* u *belongs to* $\mathrm{Dom}\delta$*. Then,*

$$d_{TV}(F, Z) \leq 2\sqrt{\mathrm{Var}(\langle DF, u \rangle_{\mathfrak{H}})},$$

where Z *is* $N(0, 1)$*.*

Stein meets Malliavin

Let $W = \{W(h), h \in \mathfrak{H}\}$ be an isonormal Gaussian process defined on $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is generated by W.

Theorem (Nourdin-Peccati)

Suppose that $F \in \mathbb{D}^{1,2}$ satisfies $E[F^2] = 1$ and $F = \delta(u)$, where u belongs to $\mathrm{Dom}\delta$. Then,

$$d_{TV}(F, Z) \leq 2\sqrt{\mathrm{Var}(\langle DF, u\rangle_{\mathfrak{H}})},$$

where Z is $N(0,1)$.

Proof: Fix $\phi \in \mathcal{C}^1(\mathbb{R})$ such that $\|\phi\|_\infty \leq \sqrt{\frac{\pi}{2}}$ and $\|\phi'\|_\infty \leq 2$. Then,

$$E[F\phi(F)] = E[\delta(u)\phi(F)] = E[\langle u, D[\phi(F)]\rangle_{\mathfrak{H}}] = E[\phi'(F)\langle u, DF\rangle_{\mathfrak{H}}].$$

Therefore,

$$\begin{aligned} |E[\phi'(F)] - E(F\phi(F))| &= |E[\phi'(F)[1 - \langle DF, u\rangle_{\mathfrak{H}}]| \\ &\leq 2E[|1 - \langle DF, u\rangle_{\mathfrak{H}}|] \\ &\leq 2\sqrt{\mathrm{Var}(\langle DF, u\rangle_{\mathfrak{H}})}, \end{aligned}$$

because $E[\langle DF, u\rangle_{\mathfrak{H}}] = E[F\delta(u)] = E[F^2] = 1$.

Proof of the estimate $d_{TV}\left(\frac{\mathcal{S}_{R,t}}{\sqrt{\mathrm{Var}(\mathcal{S}_{R,t})}}, Z\right) \leq c(t)R^{-\beta/2}$:

- We have

$$\begin{aligned}
\mathcal{S}_{R,t} &= \frac{1}{R^d}\int_{B_R}[u(t,x)-1]dx \\
&= \frac{1}{R^d}\int_{B_R}\int_0^t\int_{\mathbb{R}^d}\sigma(u(s,y))G_{t-s}(x-y)W(ds,dy) \\
&= \frac{1}{R^d}\int_0^t\int_{\mathbb{R}^d}\sigma(u(s,y))\left(\int_{B_R}G_{t-s}(x-y)dx\right)W(ds,dy).
\end{aligned}$$

Proof of the estimate $d_{TV}\left(\frac{\mathcal{S}_{R,t}}{\sqrt{\mathrm{Var}(\mathcal{S}_{R,t})}}, Z\right) \leq c(t)R^{-\beta/2}$:

- We have

$$\begin{aligned}\mathcal{S}_{R,t} &= \frac{1}{R^d}\int_{B_R}[u(t,x)-1]dx \\ &= \frac{1}{R^d}\int_{B_R}\int_0^t\int_{\mathbb{R}^d}\sigma(u(s,y))G_{t-s}(x-y)W(ds,dy) \\ &= \frac{1}{R^d}\int_0^t\int_{\mathbb{R}^d}\sigma(u(s,y))\left(\int_{B_R}G_{t-s}(x-y)dx\right)W(ds,dy).\end{aligned}$$

- Thus,

$$\mathcal{S}_{R,t} = \delta(v_{R,t}),$$

where

$$v_{R,t}(s,y) = \mathbf{1}_{[0,t]}(s)\frac{1}{R^d}\sigma(u(s,y))\left(\int_{B_R}G_{t-s}(x-y)dx\right).$$

- Computation of the variance of $\langle D\mathcal{S}_{R,t}, v_{R,t}\rangle_{\mathfrak{H}}$:

$$D\mathcal{S}_{R,t} = D\delta(v_{R,t}) = v_{R,t} + \delta(Dv_{R,t}).$$

Therefore,

$$\langle D\mathcal{S}_{R,t}, v_{R,t}\rangle_{\mathfrak{H}} = \|v_{R,t}\|_{\mathfrak{H}}^2 + \delta\left(\int_{\mathbb{R}_+\times\mathbb{R}^d} D_{s,y}v_{R,t} \times v_{R,t}(s,y)dyds\right)$$

and

$$\mathrm{Var}\left(\langle D\mathcal{S}_{R,t}, v_{R,t}\rangle_{\mathfrak{H}}\right) \le 2\mathrm{Var}(\|v_{R,t}\|_{\mathfrak{H}}^2) + 2E\int_0^t\int_{\mathbb{R}^d}\left|\int_0^r\int_{\mathbb{R}^d} D_{s,y}v_{R,t}(r,z)v_{R,t}(s,y)dyds\right|^2 dzdr.$$

- To complete the proof, we use Poincaré inequality and the estimate on the p-norm of derivative of the solution:

Lemma

Fix $p \geq 2$, $t > 0$ and $x \in \mathbb{R}^d$. Then, for almost all $(s, y) \in [0, t] \times \mathbb{R}^d$,

$$\|D_{s,y} u(t, x)\|_p \leq C G_{t-s}(x - y),$$

where C depends on $\beta, p, t, \mathrm{Lip}(\sigma)$ and $\sigma(0)$.

Sketch of the proof of the Lemma for $d = 2$:

- The derivative satisfies the linear SPDE

$$D_{s,y}u(t,x) = G_{t-s}(x-y)\sigma(u(s,y)) + \int_s^t \int_{\mathbb{R}^2} G_{t-r}(x-z)\sigma'(u(r,z))D_{s,y}u(r,z)W(dr,dz).$$

- We write the derivative as a series of iterated stochastic integrals

$$D_{s,y}u(t,x) = \sum_{j=0}^{\infty} T^{(j)}$$

Sketch of the proof of the Lemma for $d = 2$:

- The derivative satisfies the linear SPDE

$$D_{s,y}u(t,x) = G_{t-s}(x-y)\sigma(u(s,y)) + \int_s^t \int_{\mathbb{R}^2} G_{t-r}(x-z)\sigma'(u(r,z))D_{s,y}u(r,z)W(dr,dz).$$

- We write the derivative as a series of iterated stochastic integrals

$$D_{s,y}u(t,x) = \sum_{j=0}^{\infty} T^{(j)}$$

- Clearly, $\|T^{(0)}\|_p \le CG_{t-s}(x-y)$.

Sketch of the proof of the Lemma for $d = 2$:

- The derivative satisfies the linear SPDE

$$D_{s,y}u(t,x) = G_{t-s}(x-y)\sigma(u(s,y)) + \int_s^t \int_{\mathbb{R}^2} G_{t-r}(x-z)\sigma'(u(r,z))D_{s,y}u(r,z)W(dr,dz).$$

- We write the derivative as a series of iterated stochastic integrals

$$D_{s,y}u(t,x) = \sum_{j=0}^{\infty} T^{(j)}$$

- Clearly, $\|T^{(0)}\|_p \leq CG_{t-s}(x-y)$.
- Consider the case $j = 1$, where

$$T^{(1)} = \int_s^t \int_{\mathbb{R}^2} G_{t-r}(x-z)\sigma'(u(r,z))G_{r-s}(y-z)\sigma(u(s,y))W(dr,dz)$$

- By Burkholder-David-Gundy inequality

$$\|T^{(1)}\|_p \le C \int_s^t \int_{\mathbb{R}^4} G_{t-r}(x-z)G_{r-s}(y-z)G_{t-r}(x-z')G_{r-s}(y-z') \times |z-z'|^{-\beta} dzdz'dr.$$

- By Burkholder-David-Gundy inequality

$$
\|T^{(1)}\|_p \leq C \int_s^t \int_{\mathbb{R}^4} G_{t-r}(x-z)G_{r-s}(y-z)G_{t-r}(x-z')G_{r-s}(y-z') \times |z-z'|^{-\beta} dzdz'dr.
$$

- By the Hardy-Littlewood-Sobolev inequality

$$
\begin{aligned}
\|T^{(1)}\|_p &\leq C \int_s^t \|G_{t-r}(x-\bullet)G_{r-s}(y-\bullet)\|^2_{L^{2q}(\mathbb{R}^2)} dr \\
&= C \int_s^t \left(G^{2q}_{t-r} * G^{2q}_{r-s}\right)^{1/q} dr,
\end{aligned}
$$

where $q = \frac{2}{4-\beta}$.

- By Burkholder-David-Gundy inequality

$$\|T^{(1)}\|_p \le C \int_s^t \int_{\mathbb{R}^4} G_{t-r}(x-z)G_{r-s}(y-z)G_{t-r}(x-z')G_{r-s}(y-z') \times |z-z'|^{-\beta} dz dz' dr.$$

- By the Hardy-Littlewood-Sobolev inequality

$$\|T^{(1)}\|_p \le C \int_s^t \|G_{t-r}(x-\bullet)G_{r-s}(y-\bullet)\|^2_{L^{2q}(\mathbb{R}^2)} dr = C \int_s^t \left(G^{2q}_{t-r} * G^{2q}_{r-s} \right)^{1/q} dr,$$

where $q = \frac{2}{4-\beta}$.

- We have

$$\int_s^t \left(G^{2q}_{t-r} * G^{2q}_{r-s} \right)^{1/q} dr \le C G_{t-s}(x-y).$$

Stochastic wave equation in dimension $d = 1, 2$ with integrable covariance

- Suppose that $\gamma \in L^1(\mathbb{R})$ if $d = 1$ and $\gamma \in L^1(\mathbb{R}^2) \cap L^\ell(\mathbb{R}^2)$ for some $\ell > 1$ if $d = 2$.

Stochastic wave equation in dimension $d = 1, 2$ with integrable covariance

- Suppose that $\gamma \in L^1(\mathbb{R})$ if $d = 1$ and $\gamma \in L^1(\mathbb{R}^2) \cap L^\ell(\mathbb{R}^2)$ for some $\ell > 1$ if $d = 2$.
- Set

$$\mathcal{S}_{R,t} = \frac{1}{R^d} \int_{B_R} [u(t,x) - 1] dx.$$

Stochastic wave equation in dimension $d = 1, 2$ with integrable covariance

- Suppose that $\gamma \in L^1(\mathbb{R})$ if $d = 1$ and $\gamma \in L^1(\mathbb{R}^2) \cap L^\ell(\mathbb{R}^2)$ for some $\ell > 1$ if $d = 2$.
- Set

$$\mathcal{S}_{R,t} = \frac{1}{R^d} \int_{B_R} [u(t,x) - 1] dx.$$

- N.-Zheng, '20:

$$\boxed{d_{TV}\left(\frac{\mathcal{S}_{R,t}}{\mathrm{Var}(\mathcal{S}_{R,t})}, Z\right) \leq CR^{-d/2}}$$

Stochastic wave equation in dimension $d = 1, 2$ with integrable covariance

- Suppose that $\gamma \in L^1(\mathbb{R})$ if $d = 1$ and $\gamma \in L^1(\mathbb{R}^2) \cap L^\ell(\mathbb{R}^2)$ for some $\ell > 1$ if $d = 2$.
- Set
$$\mathcal{S}_{R,t} = \frac{1}{R^d} \int_{B_R} [u(t,x) - 1] dx.$$
- N.-Zheng, '20:
$$\boxed{d_{TV}\left(\frac{\mathcal{S}_{R,t}}{\mathrm{Var}(\mathcal{S}_{R,t})}, Z \right) \le CR^{-d/2}}$$
- Functional CLT (N.-Zheng, '20): Set $\xi(s) = E[\sigma(u(s,0))]$.
$$\left\{ R^{d/2} \mathcal{S}_{R,t}, t \ge 0 \right\} \xrightarrow{C(\mathbb{R}_+;\mathbb{R})} \{\mathcal{G}_t, t \ge 0\},$$
where $\mathcal{G}$ is a centered Gaussian process with covariace
$$E\left[\mathcal{G}_{t_1} \mathcal{G}_{t_2}\right] = |B_1| \int_{\mathbb{R}^d} \mathrm{Cov}(u(t_1,x), u(t_2,0)) dx.$$