Existence and uniqueness for the mild solution of the stochastic heat equation with non-Lipschitz drift on an unbounded spatial domain

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Stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + f(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x), \quad t > 0, \quad x \in \mathbb{R}^d, \\ u(0,x) = u_0(x) \end{cases}$$

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$$f(u_2) - f(u_1) \leq \kappa(u_2 - u_1).$$

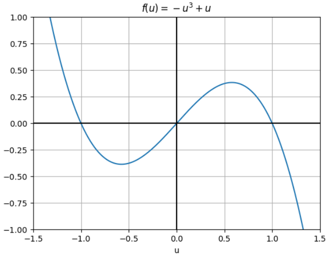
- Also assume a growth condition. There exist constants $K > 0$ and $\nu \geq 0$ such that for all $u \in \mathbb{R}$,

$$|f(u)| \leq K e^{K|u|^\nu}.$$

Examples of half-Lipschitz functions

- Odd degree polynomials with negative leading term.

$$f(u) = -\alpha u^{2m+1} + \sum_{i=1}^{2m} a_i u^i, \qquad \alpha > 0, \qquad a_i \in \mathbb{R}.$$



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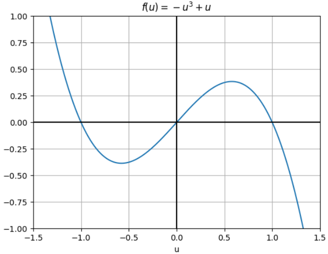

- For $u_1 < u_2$, $f(u_2) - f(u_1) \leq \kappa(u_2 - u_1)$ where $\kappa = \sup_u f'(u) < +\infty$.
- Polynomials locally Lipschitz continuous, but not globally Lipschitz continuous.

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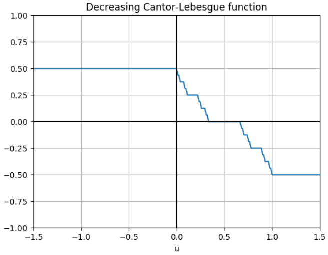

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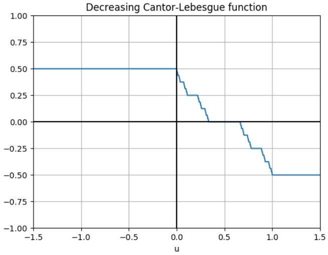

- For $u_1 < u_2$, $f(u_2) - f(u_1) \leq 0$ because it is non-increasing.
- Every half-Lipschitz function can be written as a sum $f(u) = \phi(u) + \kappa u$ where ϕ is non-increasing.

Other assumptions

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + f(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x), \quad t>0, \quad x \in \mathbb{R}^d, \\ u(0,x) = u_0(x) \end{cases}$$

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- $\sigma : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous. For any $u_2, u_1 \in \mathbb{R}$,

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  where Λ is positive and positive definite and there exists $\eta \in (0,1)$ such that its Fourier transform $\mu = \mathscr{F}(\Lambda)$ satisfies

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^{2(1-\eta)}} \mu(d\xi) < \infty.$$

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- For this talk, we assume that the initial data is bounded

$$\sup_{x \in \mathbb{R}^d} |u_0(x)| < +\infty.$$

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- The mild solution is defined to be the continuous and adapted random field solution to the integral equation

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}^d} G(t,x-y)u_0(y)dy \\ &+ \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)f(u(s,y))dyds \\ &+ \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma(u(s,y))W(dyds). \end{aligned}$$

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where $G(t,x) := (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}}$ is the fundamental solution of the heat equation.

Main result

Theorem (S. 2020)

Under these assumptions, there exists a mild solution to the stochastic heat equation that satisfies

$$\sup_{x_0 \in \mathbb{R}^d} \mathbb{E} \left| \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \frac{|u(t,x)|}{1 + |x - x_0|^\theta} \right|^p < +\infty \tag{1}$$

for any $T > 0$, $p \geq 1$ *and* $\theta > 0$.

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The mild solution is unique in the class of adapted random fields satisfying

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \mathbb{E}|\sigma(u(t,x))|^p < +\infty. \tag{2}$$

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for some $p > \frac{2(d+1)}{\eta}$ *where* η *is the constant from the strong Dalang assumption. If* σ *is bounded, then the mild solution is unique.*

Previous results - Dalang 1999

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- Dalang showed that for $p \geq 2$, a stochastic convolution can be bounded by

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \mathbb{E}\left|\int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma(s,y)W(dyds)\right|^p$$
$$\leq C_T \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \mathbb{E}|\sigma(t,x)|^p.$$

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$$\begin{aligned} u_1(t,x) \equiv& 0, \\ u_{n+1}(t,x) =& \int_{\mathbb{R}^d} G(t,x-y)u_0(y)dy \\ &+ \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)f(u_n(s,y))dyds \\ &+ \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma(u_n(s,y))W(dyds). \end{aligned}$$

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- Using Lipschitz continuity, one can show that this sequence is a contraction in the

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \mathbb{E}|u_{n+1}(t,x) - u_n(t,x)|^p$$

metric for appropriate choices of T and p.

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- $\sigma : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous with growth rate

$$|\sigma(u)| \leq K(1 + |u|^{1/m}).$$

Previous results - Cerrai 2003

- Because both f and σ were assumed to be locally Lipschitz continuous, for any $R > 0$ we can build cutoff versions f_R and σ_R such that are globally Lipschitz continuous and satisfy

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- By standard Picard iteration arguments u_R exists and is unique.
- A solution to the original equation $u(t,x) = u_R(t,x)$ for all $t \leq \tau_R$

$$\tau_R = \inf\left\{t > 0 : \sup_{x \in D} |u(t,x)| \geq R\right\}.$$

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- A priori estimates prove that $\lim_{R\to\infty} \tau_R = +\infty$ with probability one.
- Therefore, there exists a unique, global solution to the original SPDE in the bounded domain case.

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- We also do not need to assume polynomial growth.

Partial bibliography

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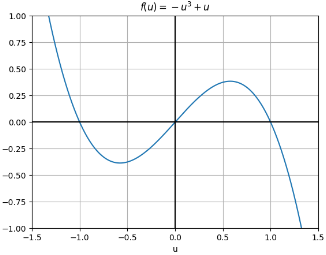

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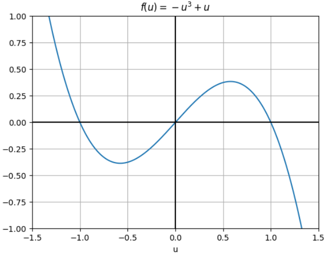

- Example: $f(u) = -u^3 + u$
- When u is very negative, $f(u)$ is very positive. When u is very positive, $f(u)$ is very negative.

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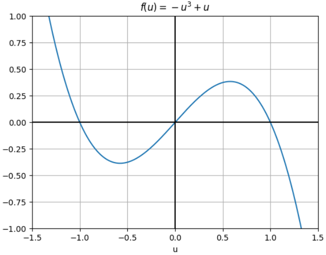

- Example: $f(u) = -u^3 + u$
- When u is very negative, $f(u)$ is very positive. When u is very positive, $f(u)$ is very negative.
- We expect this property to prevent blow-up and non-uniqueness.

The mapping $\mathcal{M}$

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- For deterministic $z(t,x)$, let $\mathcal{M}(z)(t,x)$ be the solution

$$\mathcal{M}(z)(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y) f(\mathcal{M}(z)(s,y)) dyds + z(t,x).$$

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- This means $u(t,x) = \mathcal{M}(U_0 + Z)(t,x)$.

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Weighted function spaces

- For $x_0 \in \mathbb{R}^d$ and $\theta \geq 0$, let $C_{x_0,\theta}([0,T] \times \mathbb{R}^d)$ denote the continuous functions $\psi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\lim_{|x|\to\infty} \sup_{t\in[0,T]} \frac{|\psi(t,x)|}{1+|x-x_0|^\theta} = 0,$$

endowed with the norm

$$|\psi|_{C([0,T]\times\mathbb{R}^d)} = \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \frac{|\psi(t,x)|}{1+|x-x_0|^\theta}.$$

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- We call x_0 the center of the weight.
- The $C_{x_0,\theta}$ and $C_{x_1,\theta}$ spaces coincide, but we need certain estimates to be uniform over the center.

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Theorem (S. 2020)

For any $\theta \in \left(0, \frac{2}{\nu}\right)$ there exists a constant $C = C(\kappa, \theta)$ such that $\mathcal{M}$ is a well-defined ***Lipschitz continuous*** *mapping on $C_{x_0,\theta}([0,T] \times \mathbb{R}^d)$. Specifically, for any $z_1, z_2 \in C_{x_0,\theta}([0,T] \times \mathbb{R}^d)$,*

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \frac{|\mathcal{M}(z_1)(t,x) - \mathcal{M}(z_2)(t,x)|}{1 + |x - x_0|^\theta}$$
$$\leq C e^{CT} \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \frac{|z_1(t,x) - z_2(t,x)|}{1 + |x - x_0|^\theta}.$$

SKIP PROOF

Lipschitz continuity of $\mathcal{M}$

- Let $z_1, z_2 \in C_b([0,T] \times \mathbb{R}^d)$.

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- Let $z_1, z_2 \in C_b([0,T] \times \mathbb{R}^d)$.
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- v_i are differentiable

$$\frac{\partial v_i}{\partial t}(t,x) = \frac{1}{2}\Delta v_i(t,x) + f(v_i(t,x) + z_i(t,x)).$$

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- Let $\tilde{v}(t,x) = \dfrac{(v_1(t,x) - v_2(t,x))}{\rho(x)}$. $\tilde{z}(t,x) = \dfrac{(z_1(t,x) - z_2(t,x))}{\rho(x)}$.

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- $\tilde{v}$ is differentiable

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$$\limsup_{h \downarrow 0} \frac{|\tilde{v}(t,\cdot)|_{C_0} - |\tilde{v}(t-h,\cdot)|_{C_0}}{h} \leq \frac{\partial \tilde{v}}{\partial t}(t,x_t)\text{sign}(\tilde{v}(t,x_t)).$$

Lipschitz continuity of $\mathcal{M}$

- The upper-left derivative is bounded by

$$
\begin{aligned}
&\frac{d^-}{dt}|\tilde{v}(t,\cdot)|_{C_0} \leq \frac{\partial \tilde{v}}{\partial t}(t,x_t)\text{sign}(\tilde{v}(t,x_t)) \\
&\quad \leq \frac{1}{2}\Delta\tilde{v}(t,x_t)\text{sign}(\tilde{v}(t,x_t)) \\
&\qquad + \frac{\nabla\rho}{2\rho}(x_t)\cdot\nabla\tilde{v}(t,x_t)\text{sign}(\tilde{v}(t,x_t)) \\
&\qquad + \frac{\Delta\rho}{2\rho}(x_t)\tilde{v}(t,x_t)\text{sign}(\tilde{v}(t,x_t)) \\
&\qquad + \frac{\Big(f(v_1(t,x_t)+z_1(t,x_t)) - f(v_2(t,x_t)+z_2(t,x_t))\Big)}{\rho(x_t)}\text{sign}(\tilde{v}(t,x_t)).
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\end{aligned}$$

- Because x_t is a global minimizer or maximizer, $\Delta\tilde{v}(t,x)\text{sign}(\tilde{v}(t,x_t)) \leq 0$,

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$$\begin{aligned}
\frac{d^-}{dt}|\tilde{v}(t,\cdot)|_{C_0} \leq & \frac{\partial \tilde{v}}{\partial t}(t,x_t)\operatorname{sign}(\tilde{v}(t,x_t)) \\
\leq & \frac{1}{2}\Delta\tilde{v}(t,x_t)\operatorname{sign}(\tilde{v}(t,x_t)) \\
& + \frac{\nabla\rho}{2\rho}(x_t)\cdot\nabla\tilde{v}(t,x_t)\operatorname{sign}(\tilde{v}(t,x_t)) \\
& + \frac{\Delta\rho}{2\rho}(x_t)\tilde{v}(t,x_t)\operatorname{sign}(\tilde{v}(t,x_t)) \\
& + \frac{\Big(f(v_1(t,x_t)+z_1(t,x_t)) - f(v_2(t,x_t)+z_2(t,x_t))\Big)}{\rho(x_t)}\operatorname{sign}(\tilde{v}(t,x_t)).
\end{aligned}$$

- Because x_t is a global minimizer or maximizer, $\Delta\tilde{v}(t,x)\operatorname{sign}(\tilde{v}(t,x_t)) \leq 0$, $\nabla\tilde{v}(t,x)\operatorname{sign}(\tilde{v}(t,x_t)) = 0$,

Lipschitz continuity of $\mathcal{M}$

- The upper-left derivative is bounded by

$$
\begin{aligned}
\frac{d^-}{dt}|\tilde{v}(t,\cdot)|_{C_0} &\le \frac{\partial \tilde{v}}{\partial t}(t,x_t)\mathrm{sign}(\tilde{v}(t,x_t)) \\
&\le \frac{1}{2}\Delta\tilde{v}(t,x_t)\mathrm{sign}(\tilde{v}(t,x_t)) \\
&\quad + \frac{\nabla\rho}{2\rho}(x_t)\cdot\nabla\tilde{v}(t,x_t)\mathrm{sign}(\tilde{v}(t,x_t)) \\
&\quad + \frac{\Delta\rho}{2\rho}(x_t)\tilde{v}(t,x_t)\mathrm{sign}(\tilde{v}(t,x_t)) \\
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Lipschitz continuity of $\mathcal{M}$

- To bound $\frac{d^-}{dt}|\tilde{v}(t,\cdot)|_{C_0}$, we need a bound on

$$\frac{\Big(f(v_1(t,x_t)+z_1(t,x_t)) - f(v_2(t,x_t)+z_2(t,x_t))\Big)}{\rho(x_t)}\text{sign}(\tilde{v}(t,x_t)).$$

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Lipschitz continuity of $\mathcal{M}$

- If $\text{sign}(\tilde{v}(t,x_t)) = \text{sign}(\tilde{v}(t,x_t) + \tilde{z}(t,x_t))$, then $\text{sign}(\tilde{v}(t,x_t)) = \text{sign}((v_1(t,x_t) + z_1(t,x_t)) - (v_2(t,x_t) + z_2(t,x_t)))$.

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- By half-Lipschitz assumption,

$$
\begin{aligned}
&\frac{\Big(f(v_1(t,x_t) + z_1(t,x_t)) - f(v_2(t,x_t) + z_2(t,x_t))\Big)}{\rho(x_t)}\text{sign}(\tilde{v}(t,x_t)) \\
&\leq \kappa \frac{|(v_1(t,x_t) + z_1(t,x_t)) - (v_2(t,x_t) + z_2(t,x_t))|}{\rho(x_t)} \\
&\leq \kappa |\tilde{v}(t,x_t)| + \kappa |\tilde{z}(t,x_t)| \\
&\leq \kappa |\tilde{v}(t,\cdot)|_{C_0} + \kappa |\tilde{z}(t,\cdot)|_{C_0}.
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$$\leq \kappa|\tilde{v}(t,\cdot)|_{C_0} + \kappa|\tilde{z}(t,\cdot)|_{C_0}.$$

- In this case,

$$\frac{d^-}{dt}|\tilde{v}(t,\cdot)|_{C_0} \leq C|\tilde{v}(t,\cdot)|_{C_0} + C|\tilde{z}(t,\cdot)|_{C_0}.$$

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- If $\text{sign}(\tilde{v}(t,x_t)) \neq \text{sign}(\tilde{v}(t,x_t) + \tilde{z}(t,x_t))$, then

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- Because x_t is a maximizer or minimizer of $\tilde{v}(t,\cdot)$, this implies

$$|\tilde{v}(t,\cdot)|_{C_0} = |\tilde{v}(t,x_t)| \leq |\tilde{z}(t,x_t)| \leq |\tilde{z}(t,\cdot)|_{C_0}.$$

Lipschitz continuity of $\mathcal{M}$

- These two cases show that for any $t \in [0, T]$, either

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- $|\tilde{v}(t,\cdot)|_{C_0}$ is either bounded, or it grows no faster than exponentially.
- A modified version of the Grönwall lemma shows that there exists $C > 0$ such that

$$\sup_{t\in[0,T]} |\tilde{v}(t,\cdot)|_{C_0} \leq Ce^{CT} \sup_{t\in[0,T]} |\tilde{z}(t,\cdot)|_{C_0}.$$

Lipschitz continuity of $\mathcal{M}$

- We proved that for any $z_1, z_2 \in C_b([0,T] \times \mathbb{R}^d)$,

$$\sup_{t\in[0,T]} \sup_{x \in \mathbb{R}^d} \frac{|v_1(t,x) - v_2(t,x)|}{(1 + |x - x_0|^\theta)}$$
$$\leq Ce^{CT} \sup_{t\in[0,T]} \sup_{x \in \mathbb{R}^d} \frac{|z_1(t,x) - z_2(t,x)|}{(1 + |x - x_0|^\theta)}$$

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- $\mathcal{M}(z_i) = v_i + z_i$ so

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \frac{|\mathcal{M}(z_1)(t,x) - \mathcal{M}(z_2)(t,x)|}{(1+|x-x_0|^\theta)} \leq Ce^{CT} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \frac{|z_1(t,x) - z_2(t,x)|}{(1+|x-x_0|^\theta)}$$

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- $\mathcal{M}$ is Lipschitz continuous in the $C_{x_0,\theta}([0,T] \times \mathbb{R}^d)$-norm. Can be extended to a mapping from $C_{x_0,\theta}([0,T] \times \mathbb{R}^d)$ to itself.

Kolmogorov continuity theorem

- We want to show that the stochastic integrals $\int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma(s,y)W(dyds)$ belong to the appropriate weighted function spaces.

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- Assume that $X : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a random field.
- Assume there exists $x_0 \in \mathbb{R}^d$, $A > 0$ and $\gamma \in (0,1)$ and $p > 1$ such that $p\gamma > d$ such that

$$\mathbb{E}|X(x_0)|^p \leq A$$
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- Then there exists a continuous version of this random field.

Kolmogorov continuity theorem

Theorem (S. 2020)

For any $\gamma \in (0,1)$ *and* $p > 1$ *satisfying* $p\gamma > d$*, and* $\theta > \frac{p\gamma}{p-d}$*, there exists* $C_{d,\gamma,p,\theta} > 0$ *such that*

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it follows that

$$\mathbb{E}\left(\sup_{x \in \mathbb{R}^d} \frac{|X(x)|}{1 + |x - x_0|^\theta}\right)^p \leq C_{d,\gamma,p,\theta} A.$$

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$$|f|^p_{W^{s,p}(D)} := \int_D |f(x)|^p dx + \int_D \int_D \frac{|f(x) - f(y)|^p}{|x-y|^{sp+d}} dx dy.$$

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$$|f|_{C^\theta(D)} := \sup_{x \in D} |f(x)| + \sup_{\substack{x,y \in D \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\theta} \leq C_{D,s,p} |f|_{W^{s,p}(D)}.$$

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- For a random field $Y : D \times \Omega \to \mathbb{R}$, and $sp > d$,

$$\mathbb{E}\left(\sup_{x \in D} |Y(x)|\right)^p \leq C \int_D \mathbb{E}|Y(x)|^p dx + \int_D \int_D \frac{\mathbb{E}|Y(x) - Y(y)|^p}{|x-y|^{sp+d}} dx dy.$$

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- If $\mathbb{E}|Y(x_0)|^p \leq A$ and $\mathbb{E}|Y(x_1) - Y(x_2)|^p \leq A|x_1 - x_2|^{\gamma p}$ with $\gamma p > d$,

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- then choose $s \in \left(\frac{d}{p}, \gamma\right)$.
- By fractional Sobolev embedding,

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{x \in D} |Y(x)|\right)^p \\
&\leq C \int_D \mathbb{E}|Y(x)|^p dx + C \int_D \int_D \frac{\mathbb{E}|Y(x) - Y(y)|^p}{|x-y|^{sp+d}} dx dy \\
&\leq CA + CA \int_D \int_D |x-y|^{\gamma p - sp - d} dx dy \\
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- In $\mathbb{R}^d$, $\int_{B(0,1)} |x|^\beta dx < +\infty$ for any $\beta > -d$.
- This argument only directly works for bounded $D \subset \mathbb{R}^d$.

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- Given $X : \mathbb{R}^d \times \Omega \to \mathbb{R}$ satisfying

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  with $\gamma p > d$, and $\theta > \frac{p\gamma}{p-d}$,
- Define $Y : B(0,1) := \{z \in \mathbb{R}^d : |z| < 1\} \to \mathbb{R}$ by

$$Y(z) = (1 - |z|)X\left(x_0 + z(1 - |z|)^{-\frac{1}{\theta}}\right).$$

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- $z \mapsto x_0 + z(1-|z|)^{-\frac{1}{\theta}}$ maps $B(0,1) \to \mathbb{R}^d$ one-to-one and onto.

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Main result

Theorem (S. 2020)

Under these assumptions, there exists a mild solution to the stochastic heat equation that satisfies

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- By the Kolmogorov theorem applied to stochastic convolutions, for $p > \frac{2(d+1)}{\eta}$, $\theta > \frac{d+1}{p-(d+1)}$

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- Trivially bound the right hand side

$$\leq C_T \sup_{x_0 \in \mathbb{R}^d} \mathbb{E} \left(\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \left| \frac{\sigma(\mathcal{M}(U_0 + Z_n)(t,x))}{1 + |x - x_0|^\theta} - \frac{\sigma(\mathcal{M}(U_0 + Z_{n-1})(t,x))}{1 + |x - x_0|^\theta} \right| \right)^p.$$

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- By choosing T_0 small enough so that $C_{T_0} e^{CT_0} < 1$, we can show this is a contraction.

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- The Z_n converge to a limit Z solving
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- By the Lipschitz continuity (and linear growth) of $\mathcal{M}$ in the $C_{x_0,\theta}([0,T] \times \mathbb{R}^d)$ norm,

$$\sup_{x_0 \in \mathbb{R}^d} \mathbb{E} \left| \sup_{t \in [0,T_0]} \sup_{x \in \mathbb{R}^d} \frac{|u(t,x)|}{1 + |x - x_0|^\theta} \right|^p < +\infty.$$

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- There exists a mild solution for all $t > 0$ and for any $T > 0$,

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- The continuity of $\mathcal{M}$ proves $u = u_2$.

Thank you