UNIFORM DOUBLING ON SU (2) AND BEYOND

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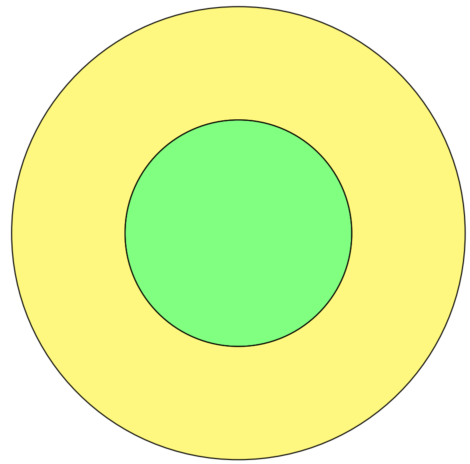
Stochastic analysis under COVID - YEAR 2020

Joint work with Nathaniel Eldredge (University of Northern Colorado) and Laurent Saloff-Coste (Cornell University)

WHAT IS UNIFORM VOLUME DOUBLING?



• Doubling: the volume of a ball of radius $2r$ can be estimated above by a constant times the volume of the concentric ball of radius r. On $\mathbb{R}^n$ the doubling constant is 2^n.

P.Diaconis and L.Saloff-Coste GAFA '94

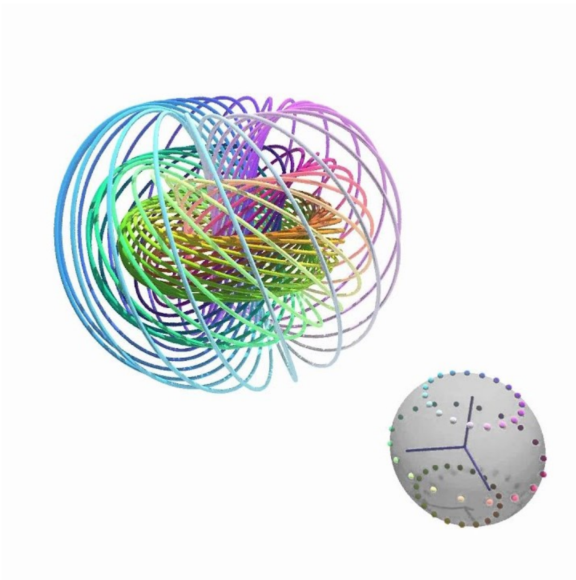
Moderate growth and random walk on finite groups

We study the rate of convergence of symmetric random walks on finite groups to the uniform distribution. A notion of moderate growth is introduced that combines with eigenvalue techniques to give sharp results. Roughly, for finite groups of moderate growth, a random walk supported on a set of generators such that the diameter of the group is r *requires order* r^2 *steps to get close to the uniform distribution... Using Gromov's theorem we show that groups with polynomial growth have moderate growth.*

Moderate growth $\frac{V(R)}{V(r)} \leqslant C\left(\frac{R}{r}\right)^d, 1 \leqslant r \leqslant R$

Doubling property $\Longrightarrow$ moderate growth

- Uniform might be in position scale *etc*
- We mean uniform over a family of metrics on a Riemannian manifold
- Local or global
- Does not hold in hyperbolic space or any fast growing volume space
- Applications to geometric analysis



Hopf fibration: `https://nilesjohnson.net/hopf.html`

GEOMETRIC ANALYSIS

(M, g) compact Riemannian manifold

$\Delta = \Delta_g$ (positive) Laplace-Beltrami operator

$\mu = \mu_g$ the Riemannian volume measure

$$\int_M u \Delta v \, d\mu_g = \int_M \langle \nabla u, \nabla v \rangle_g \, d\mu_g$$

spectrum $\lambda_0 = 0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$

$d = d_g$ diameter

$\mathrm{Vol}_g(x, r)$ $\mathrm{Vol}(x, r)$ volume function, $x \in M$, $r > 0$

heat kernel $h(t, x, y)$, $t > 0$, $x, y \in M$

$$\partial_t u + \Delta u = 0,$$

$$u(0, y) = \delta_x$$

doubling $$V(x,2r)\leqslant D_g V(x,r)$$

spectral gap $$\frac{a}{d^2}\leqslant \lambda_1\leqslant \frac{A}{d^2}$$

Weyl's eigenvalue counting

$$0<c\int_M \frac{d\mu(x)}{V\left(x,1/\sqrt{t}\right)}\leqslant \#\{i:\lambda_i<t\}\leqslant C\int_M \frac{d\mu(x)}{V\left(x,1/\sqrt{t}\right)}$$

heat kernel bounds

$$\frac{c_1}{V\left(x,\sqrt{t}\right)}e^{-\frac{c_2 d^2(x,y)}{t}}\leqslant h(t,x,y)\leqslant \frac{c_3}{V\left(x,\sqrt{t}\right)}e^{-\frac{c_4 d^2(x,y)}{t}}$$

- Parabolic Harnack inequality

- bounds on Riesz transforms

- L^1-ergodicity lower bounds

- absolute continuity of heat kernels on infinite products of compact groups

The question is how these properties/constants depend on the metric g, since otherwise the first two questions (doubling and spectral gap) are not very interesting

Example. Bounded convex domains in $\mathbb{R}^n$

$$D_{x,r,\Omega} := \frac{\mathrm{Vol}\left(B_\Omega\left(x, 2r\right)\right)}{\mathrm{Vol}\left(B_\Omega\left(x, r\right)\right)} \leqslant 2^n$$

Example. Flat tori of dimension n

Example. (P. Li, S.-T. Yau) M^n compact Riemannian manifold with non-negative Ricci curvature

Example. Bishop–Gromov comparison theorem gives a bound for D_g in terms of dimension, diameter, and Ricci curvature lower bound

(M_α, g_α) complete simply connected Riemannian manifolds

N $\quad \sup_\alpha \dim M_\alpha < \infty$

$\operatorname{diam}_\infty$ $\quad \sup_\alpha \operatorname{diam} M_\alpha < \infty$

κ $\quad \operatorname{Ric}_{g_\alpha} \geqslant -\kappa g_\alpha$

Then the doubling constant

$$D\left(N, \operatorname{diam}_\infty, \kappa\right) := \sup_\alpha D_\alpha < \infty$$

For any $x \in M$, $x_\kappa \in M_\kappa^n$

$$\varphi(r) = \frac{\operatorname{Vol} B(x, r)}{\operatorname{Vol} B(x_\kappa, r)} \searrow \qquad\qquad \varphi(r) \xrightarrow[r \to 0]{} 1$$

$$\operatorname{Vol} B\left(x, r\right) \leqslant \operatorname{Vol} B\left(x_\kappa, r\right)$$

COMPACT LIE GROUPS

K finite-dimensional) compact connected Lie group

$\mathcal{L}(K)$ all left-invariant Riemannian metrics on K

$V_g(r)$ $\mathrm{Vol}_g(B_g(r))$

Conjecture. For a compact connected Lie group K there is a constant $D(K)$ such that for any $g \in \mathcal{L}(K)$ and all $r > 0$ we have

$$\frac{V_g(2r)}{V_g(r)} \leqslant D_K$$

Consequences: uniform spectral gap, Weyl's eigenvalue counting, heat kernel bounds, Poincaré inequality, parabolic Harnack inequality etc

THE SPECIAL UNITARY GROUP $\mathbf{SU}(2)$

$\mathbf{SU}(2)$ group of 2×2 unitary matrices with determinant $+1$

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \qquad |a|^2 + |b|^2 = 1$$

Theorem. The family $\{(\mathrm{SU}(2), g) : g \in \mathcal{L}(\mathrm{SU}(2))\}$ is uniformly volume doubling.

Curvature Ricci curvatures of $\boldsymbol{g} \in \boldsymbol{\mathcal{L}}\left(\mathbf{SU}\left(\mathbf{2}\right)\right)$ can be arbitrarily negative, even for a fixed diameter. Uniform doubling does not follow from Bishop–Gromov

Geometry sub-Riemannian, collapsing

ball-box

BCDH Baker-Campbell-Dynkin-Hausdorff formula or Rodrigues formulae

MILNOR'S BASES

$g \in \mathcal{L}(K) \iff$ an inner product on the Lie algebra $\mathfrak{k}$

Lemma (Milnor 1976) For any inner product on the Lie algebra $\mathfrak{su}(2)$ there is an orthogonal basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2$$

$$g = g(a_1, a_2, a_3) \text{ with } a_i := g(e_i, e_i)$$

$$0 < a_1 \leqslant a_2 \leqslant a_3$$

cheap, moderate, expensive

BROWNIAN MOTION

$$\begin{aligned} dx_t &= \frac{1}{2}(y_t \circ dW_3 - z_t \circ dW_2) \\ dy_t &= \frac{1}{2}(z_t \circ dW_1 - x_t \circ dW_3) \\ dz_t &= \frac{1}{2}(x_t \circ dW_2 - y_t \circ dW_1) \end{aligned}$$

W_1, W_2, W_3 are independent real-valued Brownian motion with mean 0 and the covariances a_1, a_2, a_3

Denote $V_g(r) := \mathrm{Vol}_g(B_g(r))$

Theorem (Uniform volume estimates) Uniformly over $g \in \mathcal{L}(\mathrm{SU}(2))$ we have

$$V_g(r) \asymp \begin{cases} r^3, & 0 < r \leqslant \frac{a_1 a_2}{a_3} & \text{Euclidean} \\ \frac{a_3}{a_1 a_2} r^4, & \frac{a_1 a_2}{a_3} < r \leqslant a_1 & \text{Heisenberg} \\ \frac{a_1 a_3}{a_2} r^2, & a_1 < r \leqslant a_2 & \text{collapse} \\ a_1 a_2 a_3, & a_2 < r < \infty & \text{up to diameter} \end{cases}$$

$\Longrightarrow$ Uniform volume doubling

GEOMETRIC INTERPRETATION OF METRICS' PARAMETRIZATION

$a_1 \sim$ length of the shortest closed geodesic

$a_2 \sim \mathrm{diam}_g\,(\mathrm{SU}\,(2))$

$a_1 a_2 a_3 \sim V_g\,(\mathrm{SU}\,(2))$

$\mathrm{Ric}_g \gtrsim -\left(\frac{a_3}{a_1 a_2}\right)^2 g$

$a_3 \longrightarrow \infty$ approaches a sub-Riemannian geometry

$a_1 \longrightarrow 0$ collapses to S^2

SUB-RIEMANNIAN GEOMETRY ON SU (2)

$\mathcal{L}_{sub}(\mathrm{SU}(2))$	left-invariant sub-Riemannian metrics
$a_3 = \infty$	not allowed to move in e_3 direction
Hörmander's	bracket generating condition
metric	Carnot–Carathéodory distance
measure	normalized Haar measure

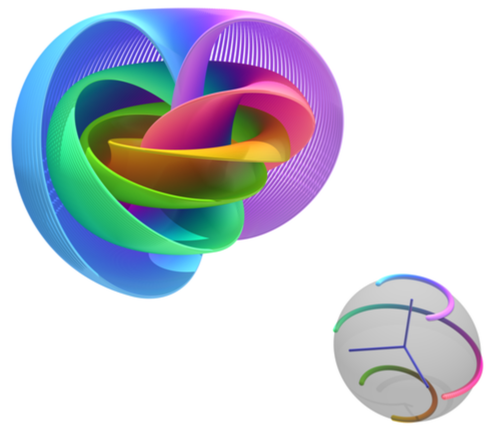
metric measure space

WEYL'S EIGENVALUE COUNTING FUNCTION ON SU (2)

$$W(t) = \#\{i : \lambda_i < t\} \asymp \begin{cases} 1, & 0 < t \leqslant \frac{1}{a_2^2} \\ a_2^2 \cdot t, & \frac{1}{a_2^2} \leqslant t \leqslant \frac{1}{a_1^2} \\ a_1^2 a_2^2 \cdot t^2, & \frac{1}{a_1^2} \leqslant t \leqslant \frac{a_3^2}{a_1^2 a_2^2} \\ a_1 a_2 a_3 \cdot t^{3/2}, & \frac{a_3^2}{a_1^2 a_2^2} \leqslant t < \infty \end{cases}$$

$$\frac{W(t)}{t^{3/2}} \xrightarrow[t\to\infty]{} a_1 a_2 a_3 \sim V_g(\mathrm{SU}(2))$$

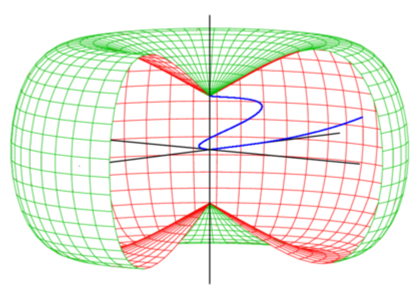
FUTURE



Other compact connected Lie groups $\mathbf{U}\,(\mathbf{2})$, $\mathbf{SU}\,(\mathbf{2}) \times \mathbb{T}^{\boldsymbol{n}}$

Measure contraction property

Sub-Riemannian and metric geometry



REFERENCES

(1) N. Eldredge, M. Gordina, L. Saloff-Coste. Left-invariant geometries on $\mathbf{SU}\,(\mathbf{2})$ are uniformly doubling, 2018, *Geometric and Functional Analysis*, 28, pp. 1321–1367.

(2) C.Judge, R.Lyons. Upper bounds for the spectral function on homogeneous spaces via volume growth, 2019, *Rev. Mat. Iberoam.*, 35, pp. 1835–1858.

(3) E. A. Lauret. On the smallest Laplace eigenvalue for naturally reductive metrics on compact simple Lie groups. 2020, *Proc. Amer. Math. Soc.* 48, pp. 3375–3380.