Stochastic Differential Graphon Games and SDEs driven by a Continuum of Independent Brownian Motions

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SAUCY2021, April 6, 2021

Credits

- **R.C., D. Cooney, C. Graves, and M. Laurière**
  Stochastic Graphon Games
  Mathematics of Operations Research 2020
- **A. Aurell, R.C., and M. Laurière**
  Linear Quadratic Stochastic Graphon Games
  in preparation

Outline

Motivation from Classical MFG Examples

- Where do **Pierre-Louis** and **Jean Michel** *put their towels on the beach?*:
  - Affinities are not *uniformly distributed*
  - One may want to avoid *some people*

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  - Fisheries interact *only* with fisheries which harvest from the same or connecting bodies of water

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- **Cournot Competition**
  - Producers compete *only* when consumer bases overlap and products are *exchangeable*

Definition of a N-player Static Network Game

- N players $\{1, 2, \cdots, N\}$
- $W = [w_{i,j}]_{i,j=1,\cdots,N}$ an $N \times N$ symmetric matrix of real numbers;
  - $w_{i,j}$ weight quantifying the strength of the **interaction** between players i and j
- A (closed convex subset of $\mathbb{R}^k$) set of admissible actions
  - $A = \mathbb{R}^k$ for the purpose of this talk
- **Strategy** profiles $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in A^N$
- Idiosyncratic **noise**, i.i.d mean-zero random variables $\xi_1, \ldots, \xi_N$ with order 2 distribution μ_0
- **State** of a player $X_{\alpha,z,\xi} = b(\alpha, z) + \xi$ when
  - taking action $\alpha \in A$
  - feeling the impact of other states through the aggregate z
  - incurring idiosyncratic random shock ξ

  where $b : A \times \mathbb{R} \ni (\alpha, z) \mapsto b(\alpha, z) \in \mathbb{R}$ is Lipschitz w.r.t. (α, z), with Lipschitz constants $c_\alpha > 0$ and $c_z \in (0, 1)$.
- **Interation** through **Aggregate**:

$$\mathbf{Z}_N\boldsymbol{\alpha} = \left(\frac{1}{N} \sum_{j=1}^{N} w_{i,j} X_{\alpha_j, z_j, \xi_j} \right)_{i=1,\ldots,N}, \qquad \boldsymbol{\alpha} \in A^N \tag{1}$$

- **Costs** + notions of **Equilibrium**

Stochastic Graphon Game: limit $N \to \infty$

Assumption: Graph $W = [w_{i,j}]_{i,j=1,\cdots,N}$ underpinning the interactions is **dense** and **converges** as $N \to \infty$. **Graphon** w

- Set of players $I = [0,1]$, $\mathcal{B}(I)$ its Borel σ-field, λ_I Lebesgue measure on $\big(I, \mathcal{B}(I)\big)$.
- Players **interact** through a **graphon** w
  (symmetric real-valued Lebesgue-measurable square-integrable function on $[0,1] \times [0,1]$).
- Set A of individual actions same as before.
- A **strategy profile** is a function $\boldsymbol{\alpha}$ which associates to each player $x \in I$ an action $\boldsymbol{\alpha}(x) = \alpha_x$ in $A \subset \mathbb{R}^k$.
- $\boldsymbol{\alpha}$ is admissible ($\boldsymbol{\alpha} \in \mathbb{A}$) if $\boldsymbol{\alpha} \in L^2(I; \mathbb{R}^k) = L^2(I, \mathcal{B}(I), \lambda_I; \mathbb{R}^k)$.
- **Idiosyncratic noise:**
  - $\boldsymbol{\xi} = (\xi_x)_{x \in I}$ is **i.i.d.**
  - **State Equation** $X_{\alpha,z,\xi} = b(\alpha, z) + \xi$ with aggregate

$$[Z\boldsymbol{\alpha}]_x = \int_I w(x,y) X_{\alpha_y,[Z\boldsymbol{\alpha}]_y,\xi_y} \lambda_I(dy), \qquad x \in I$$

Obvious Questions

- **Shouldn't $x \mapsto \xi_x$ be measurable?**
- **Shouldn't the aggregate $[Z\boldsymbol{\alpha}]_x$ be deterministic?** (LLN)

Aside: Connections with MFGs

- **Constant Graphon**
$$\exists a \in [0,1], \forall x, y \in [0,1], \quad w(x,y) = a$$
If
$$X_{\alpha,z,\xi} = b(\alpha, z) + \xi, \qquad \mathcal{J}(\alpha, z) = \mathbb{E}\left[f(X_{\alpha,z,\xi}, \alpha, z)\right],$$
Graphon Game = MFG

- **Constant Connection Strength Graphon**
$$\exists a \in [0,1], \forall x \in [0,1], \quad \int_{[0,1]} w(x,y)dy = a$$
If the aggregate $\mathbf{Z}$ is well defined, and if there exists a Nash equilibrium $\hat{\alpha}$ for the above MFG with constant a then $\hat{\boldsymbol{\alpha}}(\cdot) = \hat{\alpha}$ is a Nash equilibrium for the graphon game.

- **Piecewise Constant Graphon** games = Multi-Population MFGs

Equilibrium Theory in Macro-Economics & Finance

Huge literature on economic models with *continuum of infinitesimally small firms or agents* e.g.

J. Geanakoplos, I. Karatzas, M. Shubik, W. D. Sudderth
Inflationary equilibrium in a stochastic economy with independent agents
Journal of Mathematical Economics 52 (2014) 1–11

- *" ... we consider an economy in which a continuum of agents are subject to idiosyncratic, independent and identically distributed random shocks to their endowments."*
- *" ... we assume that the integrals*

$$\int_I Y_n^\alpha(\omega)d\alpha = Q_n(\omega) = Q$$

are constant across time-periods $n \in \mathbb{N}$ and states $\omega \in \Omega$. We assume further that the random endowments $Y_n(\alpha, \omega)$ are jointly measurable in (α, ω), where the variable α ranges over the index set $I = [0, 1]$ and the variable ω ranges over the probability space Ω ...... A consequence of our assumptions about the random endowments is that

$$\int_\Omega Y_n(\alpha, \omega)\mathbb{P}(d\omega) = \int_I Y_n(\alpha, \omega)d\alpha = Q."$$

Exact Law of Large Numbers

- **T.F. Bewley** Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers. Essays in Honor. of Girard Debreu (W. Hildenbrand and A. Mas-Colell, Eds.), (1986) pp. 79-102.
- *K.L. Judd* The law of large numbers with a continuum of IID random variables, J. Econ. Theory 35 (1985) 19–25.
- *M. Feldman, C. Gilles* An expository note on individual risk without aggregate uncertainty, J. Econ. Theory 35 (1985) 26–32.
- *Y.N. Sun* A theory of hyperfinite processes: the complete removal of individual uncertainty via exact LLN, J. Math. Econ. 29 (1998) 419–503.

J. Miao
Competitive equilibria of economies with a continuum of consumers and aggregate shocks
Journal of Economic Theory 128 (2006) 274 – 298

- *".... there are subtle technical problems, pointed out by Judd [20], associated with an environment that has a continuum of agents, e.g., measurability and the law of large numbers."*
- *" .... there is a difficulty associated with the presence of aggregate shocks. When they are present, aggregate distributions are generally random measures that may be correlated with individual shocks."*

Particle Systems Limits (without Optimization)

Given

- $\{B_x(t)\}_{t\geq 0, x\in I}$ **i.i.d.** such that $\{B_x(t)\}_{t\geq 0}$ **is a Brownian motion for each** $x \in I$
- $\{X_x(0)\}_{x\in I}$ **i.i.d. with common distribution** μ_0
- $(b,\sigma) : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}^2$ **Lip-1**
- $([w^n_{i,j}]_{i,j=1,\cdots,n})_{n\geq 0}$ sequence of interaction weights in $[0,1]$ for n particles

consider the n-particle SDE for $\big([X^n_i(t)]_{i=1,\cdots,n}\big)_{t\geq 0}$

$$X^n_i(t) = X_{i/n}(0) + \int_0^t \frac{1}{n}\sum_{j=1}^n w^n_{i,j} b\big(X^n_i(s), X^n_j(s)\big)\,ds + \int_0^t \frac{1}{n}\sum_{j=1}^n w^n_{i,j}\sigma\big(X^n_i(s), X^n_j(s)\big)\,dB_{i/n}(s),$$

Assume that the graphon W^n defined by

$$W^n\Big(\frac{i}{n},\frac{j}{n}\Big) = w^n_{i,j}, \quad \text{and} \quad W^n(x,y) = W^n\Big(\frac{\lceil nx\rceil}{n},\frac{\lceil ny\rceil}{n}\Big), \qquad x,y\in I$$

converges in the cut metric $W^n \to W$.

Then for each $0 < T < \infty$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \mathbb{E}\Big[\sup_{0\leq t\leq T}|X^n_i(t) - X_{i/n}|^2\Big] = 0$$

where, with the notation $\mu_{x,t} = \mathcal{L}(X_x(t))$,

$$\begin{aligned} X_x(t) \quad &= \quad X_x(0) + \int_0^t\int_I\int_{\mathbb{R}} w(x,y)b(X_x(s),z)\mu_{y,s}(dz)dy\; ds \\ &\qquad + \int_0^t\int_I\int_{\mathbb{R}} w(x,y)\sigma(X_x(s),z)\mu_{y,s}(dz)dy\; dB_x(s), \qquad x\in I. \end{aligned}$$

Remarks on the Previous Limit

Announced / claimed / proved by ...... in several recent arXiv announcements

- **G. Bet, F. Coppini, F. R Nardi**
  Weakly interacting oscillators on dense random graphs, arXiv
- **P. E. Caines, M. Huang**
  Graphon mean field games and the GMFG equations
- **E. Bayraktar, S. Chakraborty, R. Wu**
  Graphon Mean Field Systems

Concerning the equation in red:

1. It requires the **joint measurability** of $I \times [0, \infty) \ni (x, t) \mapsto \mu_{x,t} \in \mathcal{P}(\mathbb{R})$
2. $\big(X_x(t)\big)_{t \geq 0}$ and $\big(X_y(t)\big)_{t \geq 0}$ are **independent** if $x \neq y$ in I
3. It does not require the **joint measurability** of
   $I \times [0, \infty) \times \Omega \ni (x, t, \omega) \mapsto X_x(t, \omega)$

This result could be viewed as

◇ a form of **propagation of chaos**

◇ a justification of the 2nd approach suggested by **Feldman-Gilles**

Fubini's Extensions

Definition

A probability space $(\Omega \times I, \mathcal{W}, \mathbb{Q})$ extending the usual product space $(\Omega \times I, \mathcal{F} \otimes \mathcal{I}, \mathbb{P} \otimes \lambda)$ is said to be a **Fubini extension** of $(\Omega \times I, \mathcal{F} \otimes \mathcal{I}, \mathbb{P} \otimes \lambda)$ if for any real-valued $\mathbb{Q}$-integrable function f on $(\Omega \times I, \mathcal{W})$

(I) the two functions $f_x : \omega \mapsto f(\omega, x)$ and $f_\omega : x \mapsto f(\omega, x)$ are integrable, respectively, on $(\Omega, \mathcal{F}, \mathbb{P})$ for λ-a.e. $x \in I$, and on $(I, \mathcal{I}, \lambda)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

(II) $\int_\Omega f_x(\omega) d\mathbb{P}$ and $\int_I f_\omega(x) \lambda(dx)$ are integrable, respectively, on $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, \mathbb{P})$, with

$$\int_{\Omega \times I} f(\omega, x) d\mathbb{Q}(\omega, x) = \int_I \left(\int_\Omega f_x(\omega) d\mathbb{P}(\omega) \right) d\lambda(x) = \int_\Omega (f_\omega(x) d\lambda(x))\, d\mathbb{P}(\omega)$$

Theorem (**Sun**)

Let I be the **unit interval** and S a **Polish space**. There exist

- a probability space $(I, \mathcal{I}, \lambda)$ extending $(I, \mathcal{B}_I, \lambda_I)$
- a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a Fubini extension $(\Omega \times I, \mathcal{F} \boxtimes \mathcal{I}, \mathbb{P} \boxtimes \lambda)$ of $(\Omega \times I, \mathcal{F} \otimes \mathcal{I}, \mathbb{P} \otimes \lambda)$ such that
- for any measurable mapping $\mu : (I, \mathcal{I}) \mapsto \mathcal{P}(S)$, there is an $\mathcal{F} \boxtimes \mathcal{I}$-measurable process $\xi : \Omega \times I \to S$ such that
  - the random variables $\xi_x = \xi(\cdot, x)$ are **essentially pairwise independent**
  - $\mathbb{P} \circ \xi_x^{-1} = \mu_x$ for λ-a.e. $x \in I$

Brute Force Approach based on Fubini's Extensions

- $I = [0, 1]$, $\mathcal{B}_I$ Borel σ-field, λ_I Lebesgue measure.
- $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbb{P})$ **rich Fubini extension**
- μ_0 Wiener measure on $E = C([0, T])$
- $(\xi_x)_{x \in I}$ **essentially pairwise independent** in E with law $\mu_x = \mu_0$ for $x \in I$.
- For $t \in [0, T]$, C_t the coordinate map $E \ni \omega \mapsto C_t(\omega) = \omega(t)$. i.e. $(C_t)_{0 \leq t \leq T}$ is a Brownian motion $(E, \mathcal{B}_E, \mu_0)$
- For $t \geq 0$, and $x \in I$, $B_x(t) = C_t(\xi_x)$
- For each $x \in I$, $\mathbf{B}_x = (B_x(t))_{0 \leq t \leq T}$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$
- $(\mathbf{B}_x)_{x \in I}$ are **essentially pairwise independent** i.e. for λ-almost every $x \in I$ the process $\mathbf{B}_x$ is independent of the process $\mathbf{B}_y$ for λ-almost every $y \in I$.
- For each $x \in I$, $\mathcal{F}_t^x = \sigma\{B_x(s);\ 0 \leq s \leq t\}$

An admissible strategy profile $\alpha \in \mathbb{A}$ is a measurable and square integrable process on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbb{P})$ with values in $L^2([0, T], dt)$ which is progressive.

- w **graphon**, $\mathbf{W}$ **associated operator** on $L^2(I, \mathcal{B}_I, \lambda_I)$

NB: $\mathbf{W}$ acts on $L^2(I, \mathcal{I}, \lambda)$ since $\int_I w(x, y) f(y) \lambda(dy)$ makes sense for $f \in L^2(I, \mathcal{I}, \lambda)$.

Wishful Thinking

State Dynamics

$$dX_t^x = b(\alpha_t^x, z_t^x)dt + dB_x(t), \qquad x \in I$$

where z_t^x is the **aggregate felt by** $x \in I$ **at time** t. Intuitively:

$$z_t^x = \int_I w(x,y)X_t^y \ \lambda(dy)$$

Costs

$$J^x(\boldsymbol{\alpha}, (\hat{\boldsymbol{\alpha}}^x)^{x \in I}) = \mathbb{E}\Big[\int_0^T f(\alpha_t, , [Z\hat{\boldsymbol{\alpha}}]_t^x)dt\Big], \qquad x \in I,$$

with

- $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$ **strategy** of player $x \in I$
- $(\hat{\boldsymbol{\alpha}}^x)^{x \in I} = (\hat{\alpha}_t^x)_{0 \le t \le T}^{x \in I}$ **strategy profile** of all the players
- **aggregate** $[Z\hat{\boldsymbol{\alpha}}]_t^x = \int_I w(x,y)X_t^y \ \lambda(dy)$

Exact Law of Large Numbers (**Sun, Duffie-Sun**)i n the absence of *common* **aggregate noise**

the aggregates should be deterministic

Search for Nash Equilibria

- Start from $(\hat{\boldsymbol{\alpha}}^x)^{x\in I}$
- Solve for $(X_t^x)_{0\le t\le T}^{x\in I}$ and $([Z\hat{\boldsymbol{\alpha}}]_t^x)_{0\le t\le T}^{x\in I}$
- For a.e. $x \in I$ find $(\alpha_t^{*,x})_{0\le t\le T} = \arg\inf_{\boldsymbol{\alpha}} J^x\big(\boldsymbol{\alpha}, ([Z\hat{\boldsymbol{\alpha}}]_t^x)_{0\le t\le T}^{x\in I}\big)$
- Check that $(\hat{\boldsymbol{\alpha}}^x)^{x\in I} = (\boldsymbol{\alpha}^{*x})^{x\in I}$

Same functional analytic tools should work under appropriate assumptions

- Forward / Backward HJB / Kolmogorov system? **Caines-Huang**?
- Pontryagin? Infinite dimensional FBSDEs?

Linear Quadratic Stochastic Graphon Games

Assumptions

(I) Initial conditions $X_x(0) \in L^2(\Omega \times I; \mathbb{R})$, **e.p.i.**

(II) The functions $a, b, c : I \to \mathbb{R}$ are **$\mathcal{I}$-measurable and bounded**

Theorem

For a fixed admissible strategy profile $\boldsymbol{\alpha} \in \mathbb{A}$, there **exists a unique solution** defined for all $x \in I$ to the graphon SDE system

$$dX_t^{\boldsymbol{\alpha},x} = (a(x)X_t^{\boldsymbol{\alpha},x} + b(x)\alpha_t^x + c(x)Z_t^{\boldsymbol{\alpha},x})\, dt + dB_x(t), \;\; t \in [0, T],$$

with $X_0^{\boldsymbol{\alpha},x} = X_x(0)$,, where $Z^{\boldsymbol{\alpha},x}$ is the following **graphon aggregate**

$$Z_t^{\boldsymbol{\alpha},x} := \int_I w(x,y)X_t^{\boldsymbol{\alpha},y}\lambda(dy) = [\mathbf{W}X_t^{\boldsymbol{\alpha},\cdot}](x), \quad t \in [0, T].$$

Moreover, $Z_t^{\boldsymbol{\alpha},x}$ is **deterministic** !

Proof Strategy

Step 1.
If $\mathbf{z} \in L^2_{\boxtimes}(\Omega \times I; E)$ and $\boldsymbol{\alpha} \in \mathbb{A}$, then the **stochastic integral equation**

$$X_t^{\boldsymbol{\alpha},\cdot,z} = X_0^{\boldsymbol{\alpha},\cdot,z} + \int_0^t \left(a(\cdot)X_s^{\boldsymbol{\alpha},\cdot,z} + b(\cdot)\alpha_s^{\cdot} + c(\cdot)z_s(\cdot)\right) ds + B_{\cdot}(t), \ \ t \in [0,T], \quad X_0^{\underline{\alpha},\cdot,\mathbf{z}} = \xi^{\cdot}$$

has a **unique solution** $X^{\boldsymbol{\alpha},z} = (X_t^{\boldsymbol{\alpha},\cdot,z})_{0\le t\le T}$ in $L^2_{\boxtimes}(\Omega \times I; E)$.

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Step 2.
For each $\boldsymbol{\alpha} \in \mathbb{A}$, introduce the following mapping:

$$U^{\boldsymbol{\alpha}} : L^2_{\boxtimes}(\Omega \times I; E) \to L^2_{\boxtimes}(\Omega \times I; E),$$
$$z \mapsto U^{\boldsymbol{\alpha}} z : (\omega, x) \mapsto \left(\int_I w(x,y) X_t^{\underline{\alpha},y,z}(\omega)\lambda(dy)\right)_{t \in [0,T]}$$

and show that it is a **strict contraction**

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Proposition

For each **admissible strategy profile** $\boldsymbol{\alpha} \in \mathbb{A}$, the mapping $U^{\boldsymbol{\alpha}}$ has a **unique fixed point** $\mathbf{Z}^{\boldsymbol{\alpha}}$

$$Z_t^{\boldsymbol{\alpha},x} := \int_I w(x,y) X_t^{\boldsymbol{\alpha},y} \lambda(dy) \qquad x \in I, \ \ t \in [0, T].$$

which is **deterministic** (by the exact LLN).

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which is **deterministic** (by the exact LLN).

NB: *Result still holds for Lifschitz (non-linear) drift*

Costs & Nash Equilibrium for the Stochastic Graphon Game

Expected cost to player $x \in I$ following the admissible strategy $\beta \in \mathbb{A}(x)$ while λ-a.e. other player use the strategy profile $\boldsymbol{\alpha}$ is given by:

$$J^x(\beta; \boldsymbol{\alpha}) := \mathbb{E}\Big[\int_0^T f^x\big(X_t^{(\boldsymbol{\alpha}^{-x},\beta),x}, \beta_t, Z_t^{\boldsymbol{\alpha},x}\big)\,dt + h^x\big(X_T^{(\boldsymbol{\alpha}^{-x},\beta),x}, Z_T^{\boldsymbol{\alpha},x}\big)\Big],$$

where for $\boldsymbol{\alpha} \in \mathbb{A}$, $x \in I$ and $\beta \in \mathbb{A}(x)$

$$(\boldsymbol{\alpha}^{-x}, \beta)^y = \begin{cases} \boldsymbol{\alpha}^y, & \text{if } y \neq x \\ \beta, & \text{if } y = x. \end{cases}$$

Remark: $J^x(\beta; \boldsymbol{\alpha}) = \mathcal{J}^x(\hat{\alpha}^x, Z^{\hat{\boldsymbol{\alpha}},x})$ so:

Definition An **admissible strategy profile** $\hat{\boldsymbol{\alpha}}$ is a graphon game **Nash equilibrium** if it satisfies for all $x \in I$

$$\mathcal{J}^x(\hat{\alpha}^x, Z^{\hat{\boldsymbol{\alpha}},x}) \leq \mathcal{J}^x(\beta, Z^{\hat{\boldsymbol{\alpha}},x}), \quad \beta \in \mathcal{A}(x).$$

Stochastic Maximum Principle

Proposition

Assume that for each $x \in I$, the functions f^x **and** h^x **are bounded measurable** and that for all $(x, u, z) \in I \times \mathbb{R} \times \mathbb{R}$, $\chi \mapsto f^x(\chi, u, z)$ **and** $\chi \mapsto h^x(\chi, z)$ **are differentiable**. Then if it exists, a Nash equilibrium $\hat{\boldsymbol{\alpha}}$ must satisfy

$$\hat{\alpha}_t^x = \arg\inf_{u \in \mathbb{R}} H^x(t, \hat{X}_t^x, u, p_t^x), \quad \text{a.e. } t \in [0, T], \ \mathbb{P}\text{-a.s}, \tag{2}$$

for each $x \in I$, with $(\hat{X}^x, p^x, q^x)$ satisfying the Hamiltonian system (**adjoint equations**)

$$\begin{cases} d\hat{X}_t^x = \partial_p H^x(t, \hat{X}_t^x, \hat{\alpha}_t^x, p_t^x)dt + dB_x(t), & \hat{X}_0^x = \xi^x, \\ dp_t^x = -\partial_\chi H^x(t, \hat{X}_t^x, \hat{\alpha}_t^x, p_t^x)dt + q_t^x dB_x(t), & p_T^x = \partial_\chi h^x(\hat{X}_T^x, \hat{Z}_T^x), \end{cases}$$

where $H^x : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the **Hamiltonian of player** x

$$H^x(t, \chi, u, p) = f^x(\chi, u, \hat{Z}_t^x) + (a(x)\chi + b(x)u + c(x)\hat{Z}_t^x)p,$$

$\hat{Z}_t^x$ is the graphon aggregate corresponding to $\hat{X}_t^{\cdot}$.

If, in addition $(\chi, u) \mapsto f^x(\chi, u, z)$ **and** $\chi \mapsto h^x(\chi, z)$ **are convex, then any** $\boldsymbol{\alpha} \in \mathbb{A}$ **satisfying** (2) **for every** $x \in I$ **is a Nash equilibrium.**

Solution of the L-Q Stochastic Graphon Game

Assumptions

$$f^x(\mathrm{u}) = \frac{1}{2}\mathrm{u}^* C_f(x)\mathrm{u}, \quad h^x(\mathrm{v}) = \frac{1}{2}\mathrm{v}^* C_h(x)\mathrm{v}, \quad \mathrm{u} \in \mathbb{R}^3, \mathrm{v} \in \mathbb{R}^2.$$

for bounded measurable $C_f : I \to \mathbb{R}^{3\times 3}_{\mathrm{sym},\geq 0}$ and $C_h : I \to \mathbb{R}^{2\times 2}_{\mathrm{sym},\geq 0}$

If $[C_f(x)]_{22} > 0$, the unique minimizer of $\mathbb{R} \ni \alpha \mapsto H^x(t, \hat{X}^x_t, \alpha, p^x_t)$ is

$$\hat{\alpha}^x_t = \frac{1}{[C_f(x)]_{22}}\Big(- [C_f(x)]_{12}\hat{X}^x_t - [C_f(x)]_{32}\hat{Z}^x_t - b(x)p^x_t\Big).$$

and the corresponding FBSDE system reads

$$\begin{cases} d\hat{X}^x_t &= \left(\Gamma^0_X(x)\hat{X}^x_t + \Gamma^1_X(x)p^x_t + \Gamma^2_X(x)\hat{Z}^x_t\right)dt + dB^x_t, \quad X^x_0 = \xi^x, \\ dp^x_t &= -\left(\Gamma^0_p(x)\hat{X}^x_t + \Gamma^1_p(x)p^x_t + \Gamma^2_p(x)\hat{Z}^x_t\right)dt + q^x_t dB^x_t, \\ & \qquad p^x_T = [C_h(x)]_{11}\hat{X}^x_T + [C_h(x)]_{12}\hat{Z}^x_T, \quad \hat{Z}^x_t = \int_I w(x,y)\mathbb{E}[\hat{X}^y_t]\lambda(dy), \end{cases} \tag{3}$$

with

$$\Gamma^0_X(x) := a(x) - \frac{b(x)[C_f(x)]_{12}}{[C_f(x)]_{22}}, \qquad \Gamma^0_p(x) := [C_f(x)]_{11} - \frac{[C_f(x)]^2_{12}}{[C_f(x)]_{22}},$$

$$\Gamma^1_X(x) := -\frac{b^2(x)}{[C_f(x)]_{22}}, \qquad \Gamma^1_p(x) := \Gamma^0_X(x),$$

$$\Gamma^2_X(x) := c(x) - \frac{b(x)[C_f(x)]_{32}}{[C_f(x)]_{22}}, \qquad \Gamma^2_p(x) := [C_f(x)]_{13} - \frac{[C_f(x)]_{12}[C_f(x)]_{32}}{[C_f(x)]_{22}}.$$

It reduces to a **Riccati matrix system**

N-player Network Game

(I) Fix a sequence $x^\infty = (x_k)_{k=1}^\infty \in I^\infty$

We shall sample (with replacement) these sequences from $\lambda^\infty = \otimes_{k=1}^\infty \lambda$

(II) **Admissible strategies** for the N-player game: $\mathbb{A}_N(x^\infty)$ set of $\bigvee_{\ell=1}^N \mathbb{F}^{x_\ell}$-progressively measurable functions in $L^2(\Omega; L^2([0,T]))$

(III) **Expected costs** for player k who picks a control $\alpha^{k,N}$ in the N-player game and the other players pick $\alpha^{-k,N}$

$$J^{k,N}(\alpha^{k,N};\alpha^{-k,N}) = \mathbb{E}\Big[\int_0^T f^{x_k}(X_t^{k,N}, \alpha_t^{k,N}, Z_t^{k,N})dt + h^{x_k}(X_T^{k,N}, Z_T^{k,N})\Big]$$

(IV) where the **player states** $(X^{k,N})_{k=1}^N$ have the dynamics

$$dX_t^{k,N} = \big(a(x_k)X_s^{k,N} + b(x_k)\alpha_t^{k,N} + c(x_k)Z_t^{k,N}\big)dt + dB_{x_k}(t), \quad X_0^{k,N} = \xi^{x_k},$$

$$Z_t^{k,N} = \frac{1}{N}\sum_{\ell=1}^N w(x_k, x_\ell)X_t^{\ell,N}, \quad k = 1, \ldots, N, \;\; t \in [0,T].$$

Stochastic Maximum Principle for the N-player Game

Necessarily the equilibrium strategies must be

$$\hat{\alpha}_t^{k,N} = -\frac{1}{[C_f(x_k)]_{22}}\Big(b(x_k)p_t^{kk,N} + [C_f(x_k)]_{21}X_t^{k,N} + [C_f(x_k)]_{23}Z_t^{k,N}\Big),$$

and we need to solve for the **N state variables $(X^{k,N})_{k=1}^N$** and the **N^2 adjoint variables $(p^{kh},(q^{kh\ell})_{\ell=1}^N)_{k,h=1}^N$**

$$\begin{cases} dX_t^{k,N} = \Big(\Gamma_X^0(x_k)X_t^{k,N} + \Gamma_X^1(x_k)p_t^{kk,N} + \Gamma_X^2(x_k)Z_t^{k,N}\Big)dt + dB_{x_k}(t), \;\; X_0^{k,N} = \xi^{x_k}, \\ \qquad Z_t^{k,N} := \frac{1}{N}\sum_{\ell=1}^N w(x_k,x_\ell)X_t^{\ell,N}, \quad k=1,\dots,N, \;\; t\in[0,T] \\ dp_t^{kk,N} = -\Big(\Gamma_p^0(x_k)X_t^{k,N} + \Gamma_p^1(x_k)p_t^{kk,N} + \Gamma_p^2(x_k)Z_t^{k,N} \\ \qquad + \frac{1}{N}\sum_{\ell=1}^N c(x_\ell)w(x_k,x_\ell)p_t^{k\ell,N}\Big)dt + \sum_{\ell=1}^N q_t^{kk\ell,N}dB_{x_\ell}(t), \\ p_T^{kk,N} = [C_h(x_k)]_{11}X_T^{k,N} + [C_h(x_k)]_{12}Z_T^{k,N}, \\ dp_t^{kh,N} = -\frac{w(x_k,x_h)}{N}\left(\Gamma_p^3(x_k)X_t^{k,N} + \Gamma_p^4(x_k)p_t^{kk,N} + \Gamma_p^5(x_k)Z_t^{k,N}\right)dt \\ \qquad -\Big(\frac{1}{N}\sum_{\ell=1}^N c(x_\ell)w(x_h,x_\ell)p_t^{k\ell,N} + a(x_h)p_t^{kh,N}\Big)dt + \sum_{\ell=1}^N q_t^{kh\ell,N}dB_{x_\ell}(t), \\ p_T^{kh,N} = \frac{w(x_k,x_h)}{N}\left([C_h(x_k)]_{21}X_T^{k,N} + [C_h(x_k)]_{22}Z_T^{k,N}\right), \quad 1\le h\ne k\le N, \;\; t\in[0,T] \end{cases}$$

Propagation of Chaos

For each $x^\infty = (x_h)_{h\geq 1} \in I^\infty$, define $\zeta_N^{x^\infty}$ by

$$\zeta_N^{x^\infty} : [0,T] \times I \ni (t,x) \mapsto \frac{1}{N}\sum_{h=1}^{N} w(x,x_h)X_t^{x_h} - \int_I w(x,y)\mathbb{E}[X_t^y]\lambda(dy).$$

Proposition

There exists a constant $C>0$ s.t. for all $x^\infty \in I^\infty$ and $N \geq 1$,

$$\max_{1\leq k\leq N}\left(\mathbb{E}\left[\sup_{t\in[0,T]}\left(|X_t^{k,N} - X_t^{x_k}|^2 + |p_t^{kk,N} - p_t^{x_k}|^2\right)\right] + \sup_{t\in[0,T]}\mathbb{E}\left[|Z_t^{x_k,N} - Z_t^{x_k}|^2\right]\right)$$
$$\leq \sup_{(t,x)\in[0,T]\times I} C\left(\mathbb{E}\left[|\zeta_N^{x^\infty}(t,x)|^2\right] + \frac{1}{N}\right)$$

and the size of $\zeta_N^{x^\infty}$ can be controlled by a Banach space valued LIL !

For all $\varepsilon > 0$, there exists a $N_\varepsilon : I^\infty \mapsto \mathbb{N}$ such that

$$\bar{\lambda}^\infty\left[\sup_{(t,x)\in[0,T]\times I}\mathbb{E}\left[|\zeta_N^{x^\infty}(t,x)|^2\right] \leq \frac{(C+\varepsilon)^2\log\log N}{N},\ N \geq N_\varepsilon(x^\infty)\right] = 1$$

Approximation Result

Proposition

(I) For $\varepsilon_N := 2C\sqrt{N^{-1}\log\log N}$

(II) For λ^∞-a.e. $x^\infty = (x_k)_{k\geq 1} \in I^\infty$,

the **strategy profile** $(\hat{\alpha}^{x_k})_{k=1}^N$ given by **the graphon game Nash equilibrium** is an ε_N-Nash equilibrium for the N-player game between the players $(x_1, \ldots, x_N)$ when N is large enough (i.e. $N \geq \underline{N}(x^\infty)$).

In other words, for all $k = 1, \ldots, N$ and $\beta \in \mathbb{A}_N(x^k)$:

$$J^{k,N}(\hat{\alpha}^{x_k}; \hat{\boldsymbol{\alpha}}^{-k,N}) - J^{k,N}(\beta; \hat{\boldsymbol{\alpha}}^{-k,N}) \leq \varepsilon_N,$$

where $\hat{\boldsymbol{\alpha}}^{-k,N} := (\hat{\alpha}^{x_1}, \ldots, \hat{\alpha}^{x_{k-1}}, \hat{\alpha}^{x_{k+1}}, \ldots, \hat{\alpha}^{x_N})$.

This recasts this approach in one of the frameworks proposed by

Feldman-Gilles.